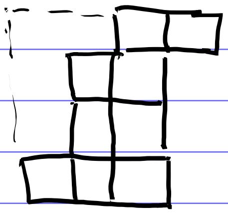
Jacobi-Trudi identities

Know: s_λ can be expressed in h_k's and in e_k's.

How to do this explicitly?

$\lambda = (\lambda_1, \ldots, \lambda_n)$, $\mu = (\mu_1, \ldots, \mu_n)$ partitions

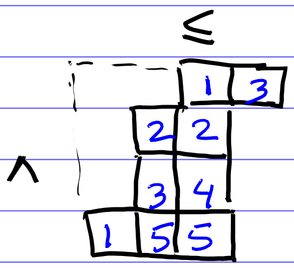
$\lambda \geq \mu$, λ/μ skew Young diagram

Ex. $\lambda/\mu =$  $\lambda = (4,3,3,3)$, $\mu = (2,1,1,0)$

Def Skew Schur function:

$$s_{\lambda/\mu} = \sum_{\substack{T:\text{ SSYT} \\ \text{of shape } \lambda/\mu}} x^{\text{weight}(T)}$$

(SSYT: filling of boxes of λ/μ by 1's, 2's, ... weakly increasing in rows, strictly increasing in cols.)

Ex. 

a semi-standard Young tableau of skew shape λ/μ

Lemma. $s_{\lambda/\mu}$ is a symmetric function.

Proof Same proof as for s_λ based on Bender-Knuth involutions. □

Jacobi-Trudi:

(1) $$s_{\lambda/\mu} = \det\left(h_{\lambda_i - i - \mu_j + j}\right)_{i,j=1}^{n}$$

(2) $$s_{\lambda/\mu} = \det\left(e_{\lambda'_i - i - \mu'_j + j}\right)_{i,j=1}^{m}$$

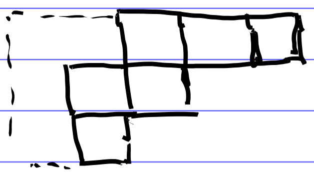
parts of conj. partition

$n \geq$ # rows of λ/μ

$m \geq$ # columns of λ/μ

$h_0 = e_0 = 1$, $h_k = e_k = 0$ for $k < 0$

Ex. $\lambda = (5,3,2)$ $\mu = (2,1,1)$

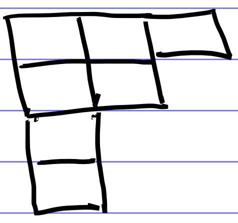
$\lambda/\mu =$ 

$$s_{\lambda/\mu} = \begin{vmatrix} h_3 & h_5 & h_6 \\ 1 & h_2 & h_3 \\ 0 & 1 & h_1 \end{vmatrix}$$

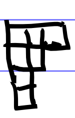
For straight shapes ($\mu = (0, \ldots, 0)$) we get

$$s_\lambda = \det\left(h_{\lambda_i - i + j}\right)_{i,j=1}^{n}$$

$$= \det\left(e_{\lambda'_i - i + j}\right)_{i,j=}^{m}$$

Ex. $\lambda =$  $= (3,2,1,1)$

$= (4,2,1)'$

s  $= \begin{vmatrix} h_3 & h_4 & h_5 & h_6 \\ h_1 & h_2 & h_3 & h_4 \\ 0 & 1 & h_1 & h_2 \\ 0 & 0 & 1 & h_1 \end{vmatrix} = \begin{vmatrix} e_4 & e_5 & e_6 \\ e_1 & e_2 & e_3 \\ 0 & 1 & e_1 \end{vmatrix}$

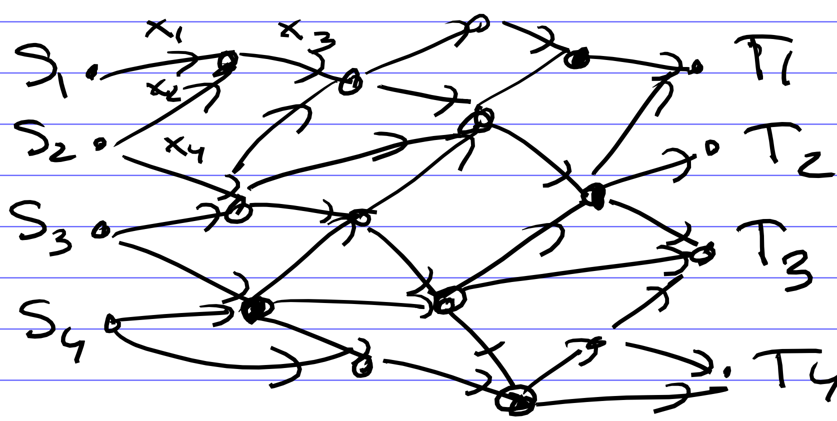
In order to prove this combinatorially ...

Gessel-Viennot Method

aka Lindström Lemma

G - planar acyclic digraph drawn on the plane with sources $S_1, \ldots, S_n$ on the left and sinks (or targets) $T_1, \ldots, T_n$ on the right ordered (ordered top-to-bottom) with weights x_e assigned to edges e.

Ex.



$$M_{ij} = \sum_{\substack{P \text{ - directed path} \\ \text{from } S_i \text{ to } T_j}} \prod_{e \in P} x_e$$

$M = (M_{ij})$ $n \times n$ matrix

Lindström Lemma

$$\det(M) = \sum_{\substack{P_1: S_1 \to T_1 \\ \vdots \\ P_n: S_n \to T_n}} \prod_{i=1}^{n} \prod_{e \in P_i} x_e$$

prod. of weight of all edges in the paths

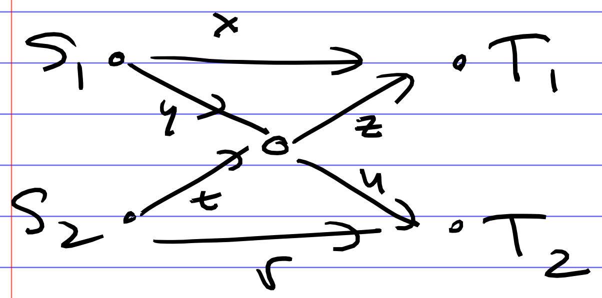
$P_1, \ldots, P_n$ - pairwise non-crossing paths

Def "non-crossing paths"

= $\forall i \neq j$, P_i & P_j have no common vertices

the paths cannot cross or even touch each other.

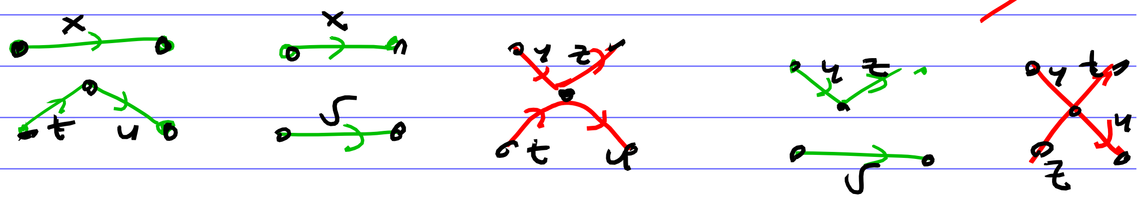
Example



$$\begin{vmatrix} x + yz & yu \\ tz & tu + s \end{vmatrix} =$$

$= (x+yz)(tu+s) - tz \cdot yu =$

$= x \cdot tu + x \cdot s + yz \cdot tu + yz \cdot s - tz \cdot yu$



Crossing pairs of paths cancel each other.

Proof (involution principle)

$$\det(M) = \sum_{w_1,\dots,w_n \text{ permutation}} (-1)^{\ell(w)} M_{1w(1)} \cdots M_{nw(n)}$$

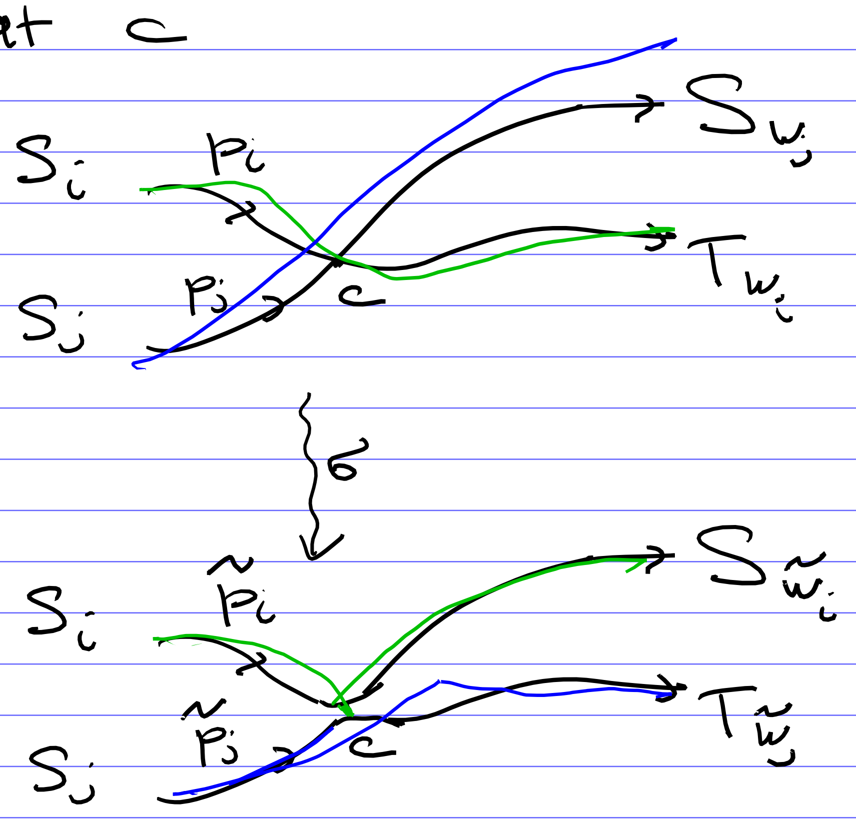
$$= \sum_{w} (-1)^{\ell(w)} \sum_{\begin{pmatrix} p_1 : S_1 \to T_{w_1} \\ \vdots \\ p_n : S_n \to T_{w_n} \end{pmatrix}} \prod_i \prod_{e \in p_i} x_e$$

arbitrary paths connecting the sources with the targets

We want to "cancel" all negative terms by constructing a <u>sign reversing</u> <u>weight preserving</u> <u>involution</u> on "bad" collections of paths.

σ - involution on collections of paths $(p_1, \dots, p_n)$ with at least one crossing ($\exists\, i \neq j$ s.t. p_i & p_j have a common vertex)

- Find the "first intersection point" c of some P_i & P_j
- "Swap the tails" of P_i & P_j at c



$\sigma : (P_1 .. P_i .. P_j .. P_n) \longmapsto (P_1 .. \tilde{P}_i .. \tilde{P}_j .. P_n)$

perm. w — perm $\tilde{w}$ obtained from w by switching w_i & w_j

sign-reversing : $\text{sign}(w) = -\text{sign}(\tilde{w})$

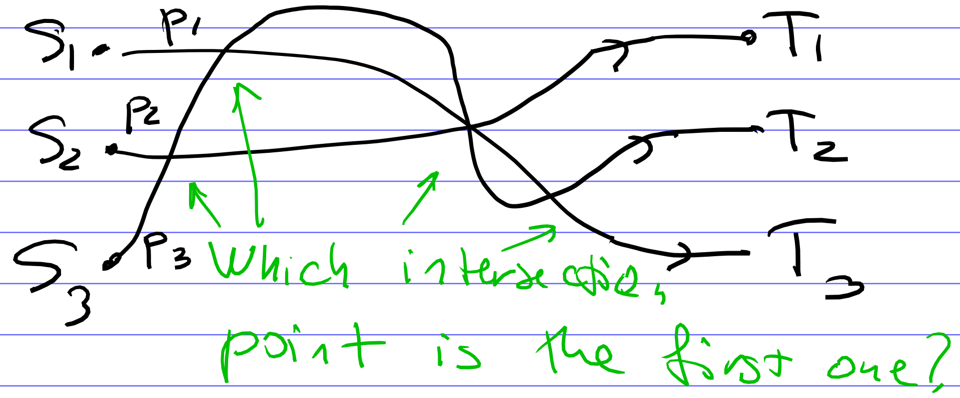
weight-preserving :

$\prod_i \prod_{e \in P_i} x_e$ does not change

The only subtle point is : What is the "first intersection point" ?

We want σ to be an involution.

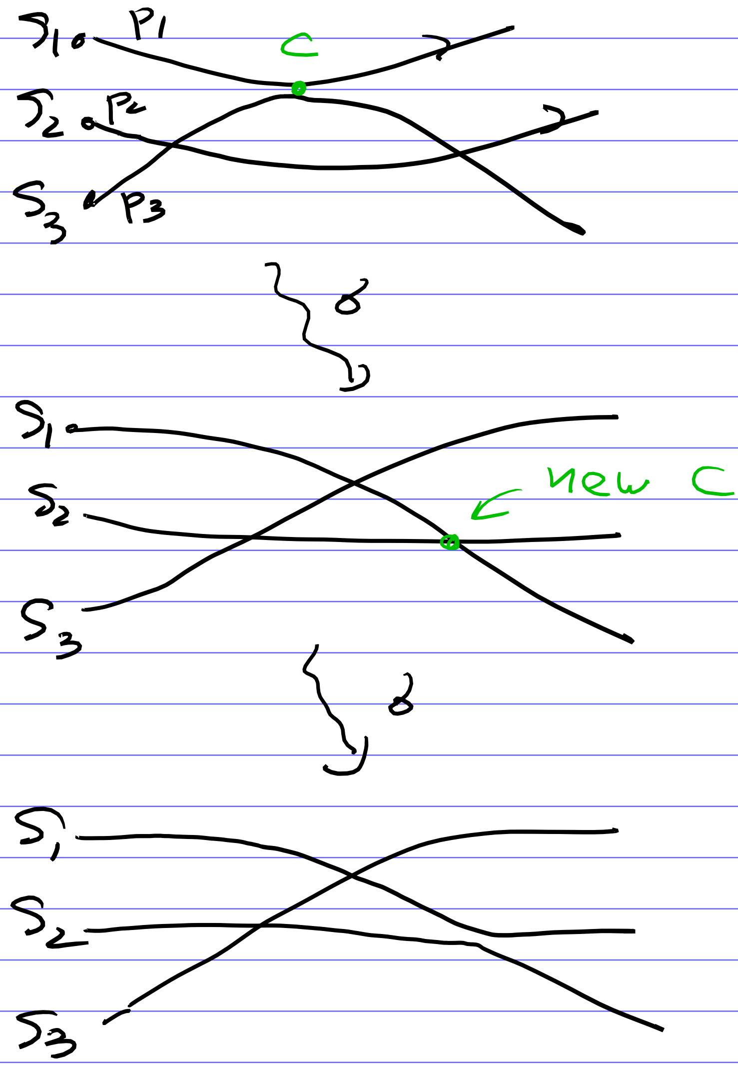
Ex.



- Find <u>lex. minimal</u> pair (i, j) s.t. p_i & p_j have a common vertex.
- Find the first common vertex of p_i & p_j

But this does not work...

<u>Couter example.</u>

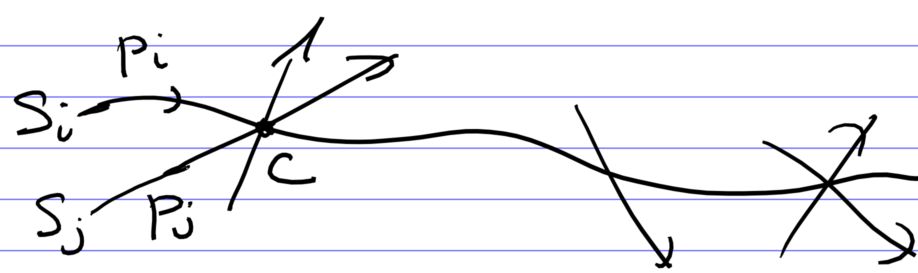


σ is <u>not</u> an involution

2nd try

- Find minimal i s.t. p_i has a common vertex with some other p_j
- Find the first vertex c of p_i that belongs to another path
- Find minimal $j \neq i$ s.t. p_j passes through c.

Ex



Then "swap the tails" of P_i & P_j at c.
to get the paths $\tilde{p}_i$ & $\tilde{p}_j$

$$\sigma : (\ldots p_i \ldots p_j \ldots) \mapsto (\ldots \tilde{p}_i \ldots \tilde{p}_j \ldots)$$

<u>Lemma</u> σ is a sign-reversing weight-preserving involution on collections of paths $(p_1, \ldots, p_n)$ with at least 1 common point of some p_i & p_j.

<u>Proof</u>. The only non-trivial thing that we need to check is that σ is an <u>involution</u>.

$(p_1, \ldots, p_n)$. Find i, c, j

$\overset{\sigma}{\rightsquigarrow} (p_1, \ldots \tilde{p}_i \ldots \tilde{p}_j \ldots p_n)$

Then $(p_1 \ldots \tilde{p}_i \ldots \tilde{p}_j \ldots p_n)$ will produce the <u>same triple</u> i, c, j.

So $(\ldots \tilde{p}_i \ldots \tilde{p}_j \ldots) \mapsto (\ldots p_i \ldots p_j \ldots)$

i.e. σ is an involution. $\square$

... back to Lindström

$$\det(M) = \sum_{\substack{\text{all collections}\\ \text{of paths}\\ (p_1,\dots,p_n)}} \pm \prod \text{weights}$$

$$= \sum_{\substack{(p_1,\dots,p_n)\\ \text{pairwise}\\ \text{non-crossing}\\ \text{collections of}\\ \text{paths}}} \prod \text{weights}$$

σ cancels all collections of paths with a crossing

□

a *minor* is det of a square submatrix

Corollary If all edge weights $x_e > 0$. Then *any minor* (of any size) of matrix M is non-negative.

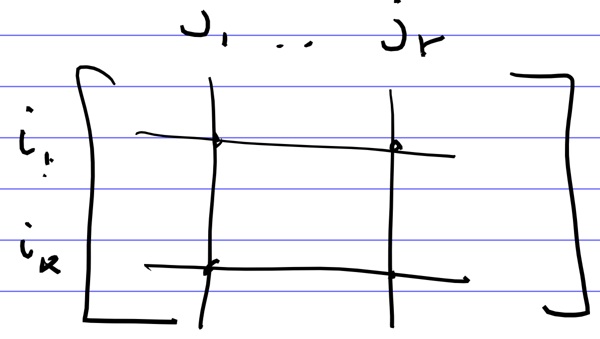
Def Such matrices are called *totally non-negative* matrices.

Ex. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $a, b, c, d \geq 0$

$ad - bc \geq 0$

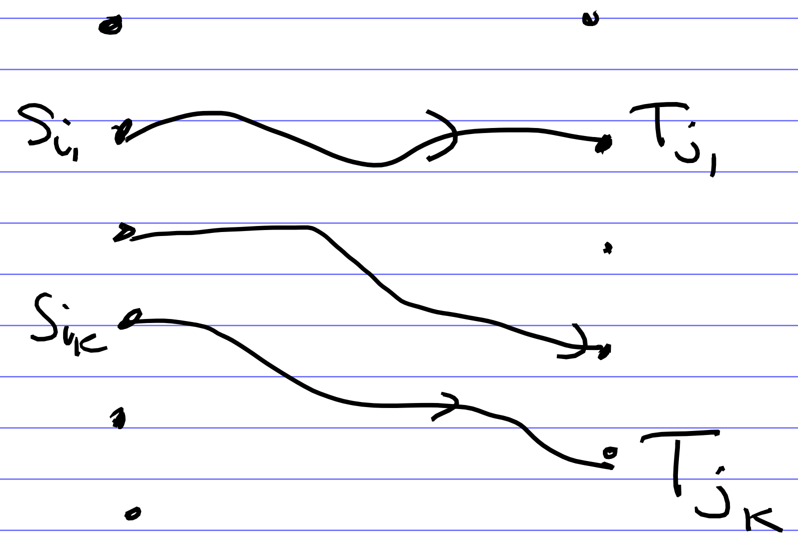
By Lindström Lemma

The Minor of M in rows $i_1 .. i_k$ and columns $j_1 .. j_k$



is

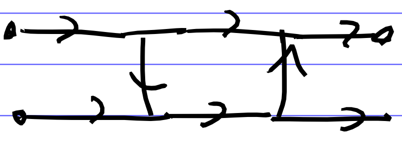
given by the sum over collections of k noncrossing paths connecting $S_{i_1} .. S_{i_k}$ with $T_{j_1} .. T_{j_k}$



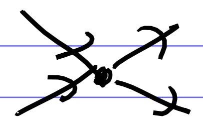
<u>Inverse Claim.</u> Any totally nonnegative matrix comes from a planar graph as above as matrix M.

<u>Example</u> $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $a, b, c, d \geq 0$, $ad - bc \geq 0$

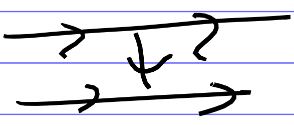
Comes from graph



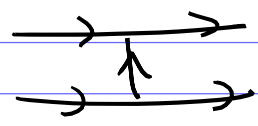
if $a, b, c, d > 0$, $ad - bc > 0$



if $a, b, c, d > 0$, $ad - bc = 0$



if $a, b, d > 0$, $c = 0$



if $a, c, d > 0$, $b = 0$

etc.

For any choice of which minors are > 0 and which minors are $= 0$, there exist a graph with positive edge weights that "realizes" the matrix.

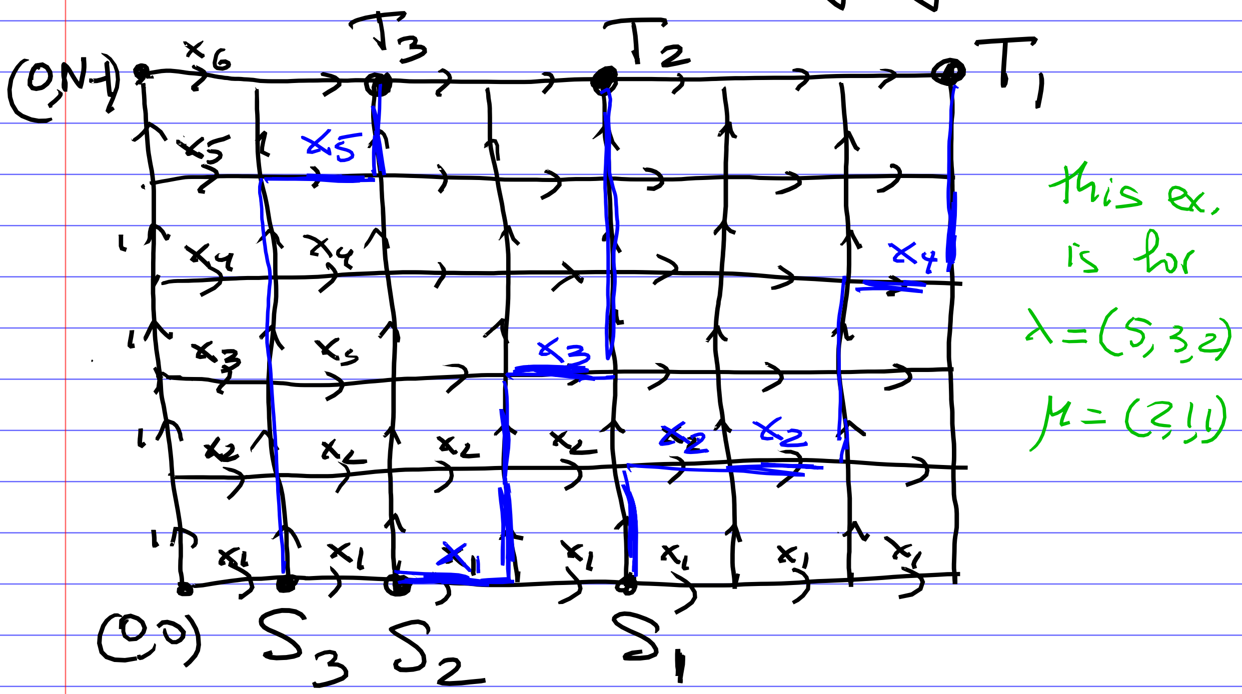
... back to Jacobi–Trudi

Let's prove it for symmetric polynomials in variables $x_1, \dots, x_N$.

$\lambda = (\lambda_1, \dots, \lambda_n)$ $\mu = (\mu_1 \dots \mu_n)$

$$S_{\lambda/\mu}(x_1, \dots, x_N) =$$
$$= \det \left(h_{\lambda_i - i - \mu_j + j}(x_1 \dots x_N) \right)_{i,j=1}^{n}$$

Consider the following graph:



- square grid of height $N-1$.
- edges directed up & right
- vertical edges have weights 1
- horizontal edges on level i have weights x_i
- sources $S_i = (\mu_i + n - i, 0)$
  targets $T_j = (\lambda_j + n - j, N-1)$

Ex. above graph

$$M_{33} = x_1 + \dots + x_6 = h_1(x_1, \dots, x_N)$$

$$M_{22} = x_1 x_1 + x_1 x_2 + \dots = h_2(x_1, \dots, x_N)$$

In general

$$M_{ij} = h_{\lambda_j - j - \mu_i + i}(x_1, \dots, x_N)$$

- # horizontal edges in a path from S_i to T_j.
- weight of edges in such paths can be any product $x_{i_1} \dots x_{i_k}$ for $1 \le i_1 \le i_2 \le \dots \le i_k \le N$

$$k = \lambda_j - j - \mu_i + i.$$

Lindström Lemma:

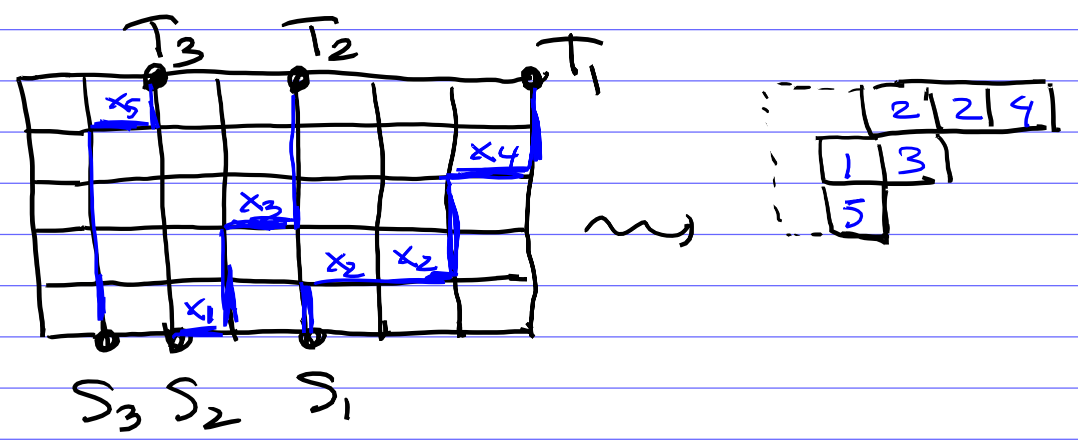
$$\det\left(h_{\lambda_j - j - \mu_i + i}\right) = \sum_{\substack{P_1 \dots P_n \\ \text{non-crossing} \\ \text{paths.}}} \prod \text{weights}$$

Lemma

$$\left\{ \begin{array}{l} \text{Collections of} \\ \text{non-crossing paths} \\ (p_1, \dots, p_n) \end{array} \right\} \overset{bij}{\longleftrightarrow} \left\{ \begin{array}{l} \text{SSYT's} \\ \text{of shape} \\ \lambda/\mu \end{array} \right\}$$

$$\prod_i \prod \text{weights of edges in } p_i = x^{\text{weight}(T)}$$

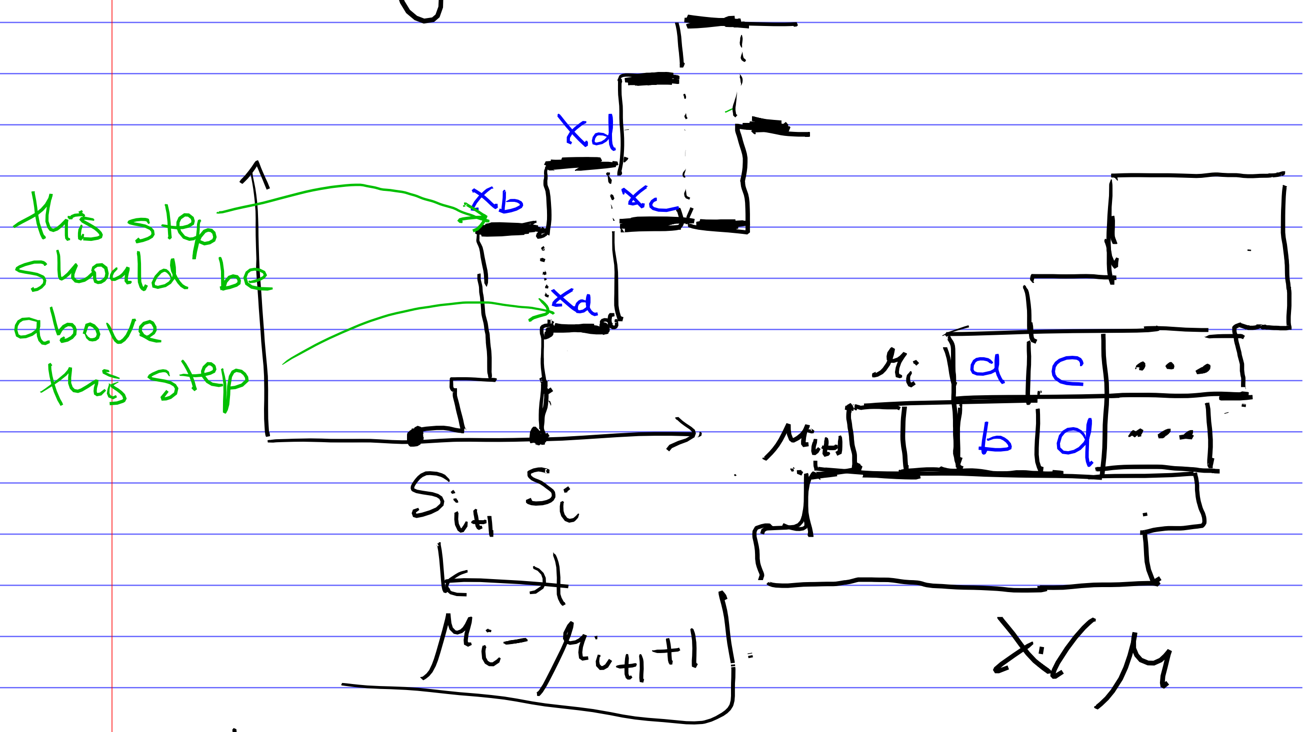
Ex. $\lambda = (5,3,2)$ $\mu = (2,1,1)$



If $p_i : S_i \to T_i$ has weight $x_{j_1} x_{j_2} \dots x_{j_k}$ (horizontal edges on levels $j_1 \le \dots \le j_k$), then put numbers $j_1, \dots, j_k$ in i-th row of shape λ/μ.

Clearly, we obtain a filling of λ/μ which is weakly increasing in rows.

Why is it strictly increasing in columns?



$a < b$, because otherwise p_{i+1} will bump into the first vertical segment of p_i

$c < d$, because otherwise p_{i+1} will bump into the next vertical segment of p_i

etc.

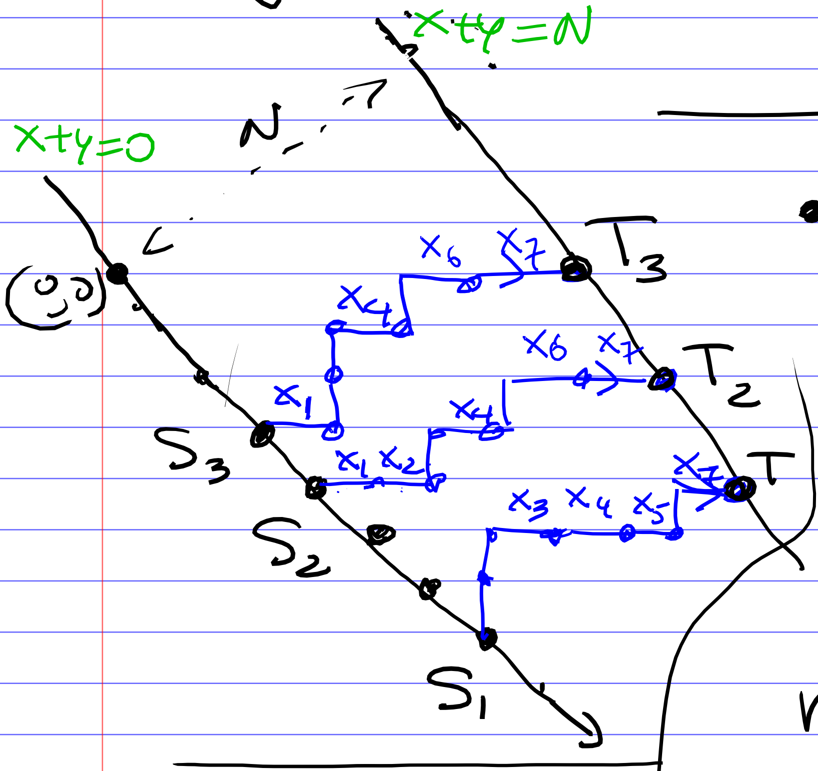
This also shows that $\forall$ SSYT $T \rightsquigarrow$ corresponding collection of paths is noncrossing.

This proves 1st JT identity

$$s_{\lambda/\mu} = \left(h_{\lambda_i - i - \mu_j + j} \right).$$

How to prove the 2nd identity $s_{\lambda/\mu} = \left(e_{\lambda'_i - i - \mu'_j + i}\right)$

Need to use a different choice for sources & targets in the square grid & different edge weights



- diagonal strip in the square grid of width N
- edges directed right & up

- weights of vertical edges = 1.
- weights of horizontal edges in i^{th} diagonal = x_i

- sources: $S_i = (\mu'_i + m - i, -(\mu'_i + m - i))$
- targets: $T_j = (\lambda'_j + m - j, N - (\lambda'_j + m - j))$

$i, j \in \{1, \dots, m\}$

Then $M = (M_{ij})$

$M_{ij} = e_{\lambda'_j - j - \mu'_i + i}(x_1, \dots, x_N)$

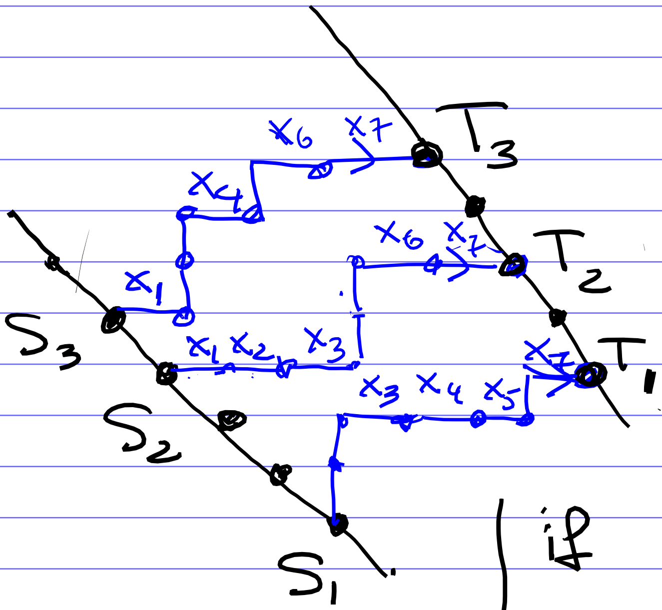
Lindström Lemma $\Rightarrow$

$\det\left(e_{\lambda'_j - j - \mu'_i + i}\right) =$

$= \sum_{\substack{\text{non-crossing} \\ \text{collection of paths}}} \prod \text{weight}$

<u>Claim</u> Such non-crossing collections $\xleftrightarrow{\text{bij}}$ SSYT's of shape λ/μ.

<u>Ex.</u>



$\lambda/\mu =$ ($\leq$ along rows, $<$ down columns)

	1	1
	2	4
3	3	6
4	6	7
5	7	
7		

if p_i has weight $x_{j_1}\cdots x_{j_k}$, $j_1 < \cdots < j_k$, then put $j_1, \dots, j_k$ in i^{th} column of λ/μ

we get a filling of λ/μ which is strictly increasing in <u>columns</u> weakly increasing in rows

$\Updownarrow$ the paths $p_1, \dots, p_n$ are non-crossing $\square$

Recall the involution ω, which is the automorphis of Λ s.t. $e_k \overset{\omega}{\longleftrightarrow} h_k \quad \forall k$

2 Jacobi-Trudi identities $\Longrightarrow$

<u>Corollary.</u> $\omega(S_{\lambda/\mu}) = S_{\lambda'/\mu'}$.