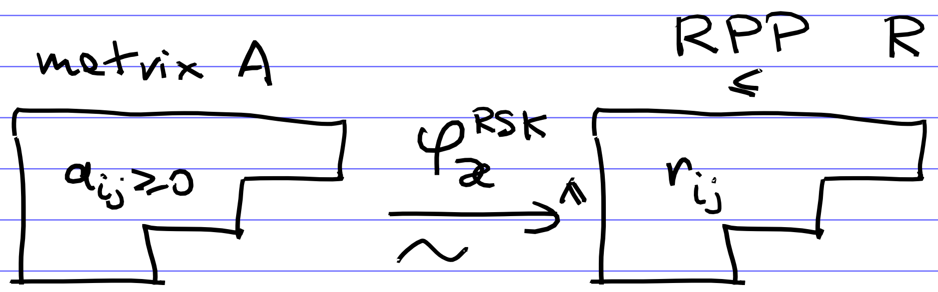
last time: "piecewise-linear RSK"

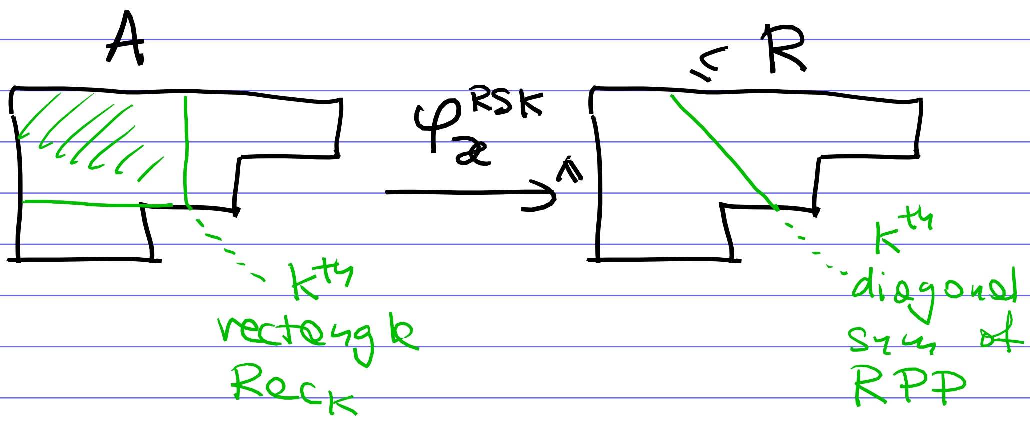
Fix shape $\varkappa$.

a bijection $\varphi_{\varkappa}^{RSK}$:



such that

k^{th} diagonal sum d_k of RPP R $=$ the sum of entries of matrix A in k^{th} rectangle Rec_k:



$$\Rightarrow \sum d_k = \sum_{ij} r_{ij} = \sum_{ij} h_{ij}\, a_{ij}$$

the sum of all enties of RPP

the sum of entries of matrix A weighted by the hooklengths h_{ij}

Example. $\varkappa = (3,3,2)$

matrix

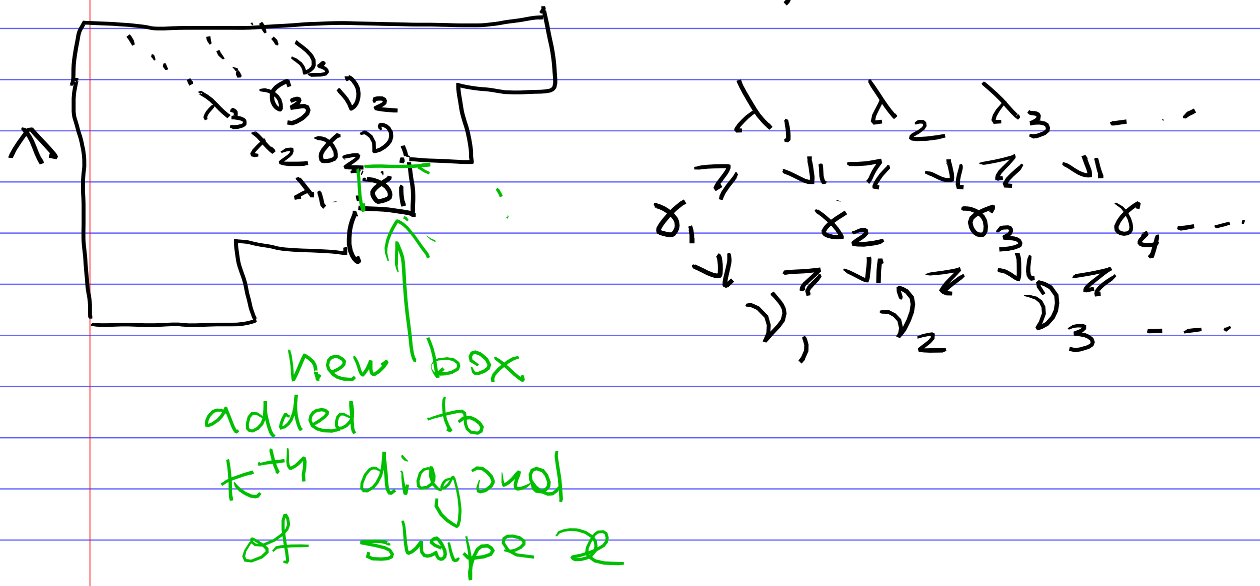
$$\begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & \end{matrix} \longleftrightarrow \begin{matrix} r_{11} & \leq & r_{12} & \leq & r_{13} \\ \wedge\!\!| & & \wedge\!\!| & & \\ r_{21} & \leq & r_{22} & \leq & r_{23} \\ \wedge\!\!| & & \wedge\!\!| & & \\ r_{31} & & r_{32} & & \end{matrix}$$

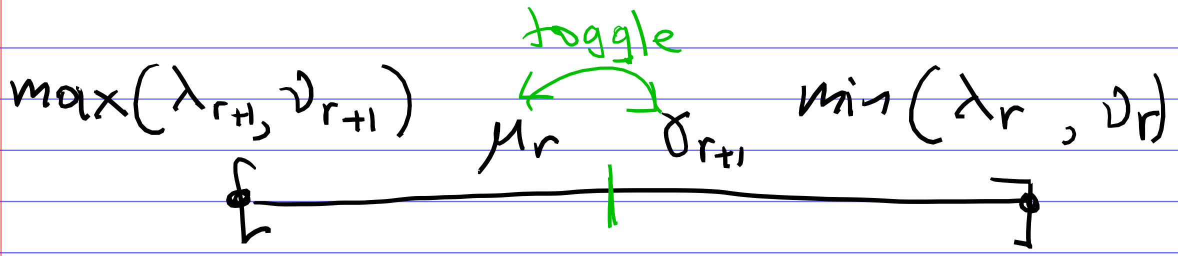
RPP

$$5a_{11} + 4a_{12} + 2a_{13} + 4a_{21} + 3a_{22} + a_{23} + 2a_{31} + a_{32} = \sum r_{ij}$$

This correspondence $\varphi_{\varkappa}^{RSK}$ was constructed inductively, starting with $\varkappa = \emptyset$ & empty RPP, by adding boxes to $\varkappa$ and using toggle operations:

When we add a box (i,j) to $\varkappa$ in k^{th} diagonal, we modify only the entries of RPP located in k^{th} diagonal, as follows:





$$\begin{cases} \gamma_{r+1} = \max(\lambda_{r+1}, \nu_{r+1}) + \min(\lambda_r, \nu_r) - \gamma_r, \quad r \geq 1 \\ \gamma_1 = \max(\lambda_1, \nu_1) + a_{ij} \end{cases}$$

the entry of matrix A in the added box

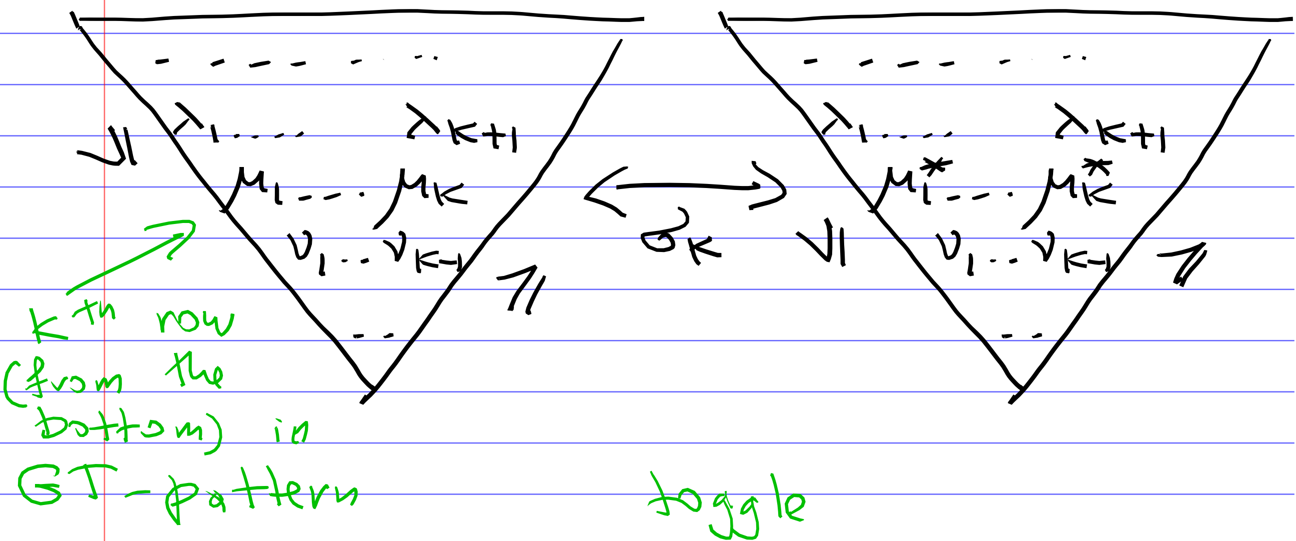
Actually, we already used toggle operations in disguise when we proved that the combinatorial definition of s_λ gives a symmetric function using

<u>Bender–Knuth involutions</u> σ_k:

$$\left\{\begin{array}{l}\text{SSYT's}\\ \text{of weight}\\ (\beta_1 \ldots \beta_k \beta_{k+1} \ldots \beta_n)\end{array}\right\} \overset{\sigma_k}{\longleftrightarrow} \left\{\begin{array}{l}\text{SSYT's}\\ \text{of weight}\\ (\beta_1 \ldots \beta_{k+1} \beta_k \ldots \beta_n)\\ \text{and the same}\\ \text{shape}\end{array}\right\}$$

β_k & β_{k+1} switched

In terms of <u>Gelfand-Tsetlin</u> patterns, the <u>Bender-Knuth involutions</u> σ_k are:



toggle

$\max(\lambda_{r+1}, \nu_r) \quad \mu_r \quad \mu_r^* \quad \min(\lambda_r, \nu_{r-1})$

$$\mu_r^* = \max(\lambda_{r+1}, \nu_r) + \min(\lambda_r, \nu_{r-1}) - \mu_r, \quad r = 1, \ldots, k$$

<u>Lemma</u>. This BK-involution σ_k switches β_k & β_{k+1} in the weight of SSYT & preserves the shape.

Hillmann - Grassl correspondence

is a bijection:

$$\left\{ \begin{array}{c} \text{matrix } A \\ \boxed{a_{ij} \geqslant 0} \\ a_{ij} \in \mathbb{Z}_{\geqslant 0} \end{array} \right\} \xleftrightarrow{\varphi^{HG}_{x}} \left\{ \begin{array}{c} \leqslant \quad \text{RPP } P \\ \geqslant \; \boxed{r_{ij}} \\ r_{ij} \in \mathbb{Z}_{\geqslant 0} \end{array} \right\}$$

between the same sets as in φ^{RSK}_{x}

that has the same properties as φ^{RSK}_{x} (the same equality between diagonal sums of RPP R and rectangular sums in matrix A)

In particular, we also have

$$\sum h_{ij} a_{ij} = \sum r_{ij}$$

But the construction of φ^{HG}_{x} is different from the construction of φ^{RSK}_{x}.

Hillman – Gressl:

$$\left\{\begin{array}{c}\text{RPP's} \\ \text{of shape } \lambda\end{array}\right\} \underset{\sim}{\longrightarrow} \left\{\begin{array}{c}\text{matrices} \\ \text{of shape } \lambda\end{array}\right\}$$

$$R \qquad\qquad\qquad\qquad A$$

Construct the ribbon path P in RPP, as follows:

(1) P starts at the rightmost box (a,b) of the first of R that has at least 1 non-zero entry.

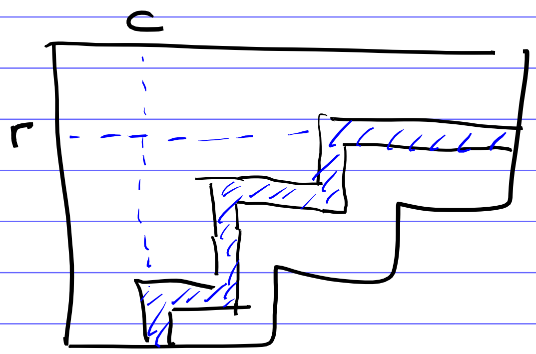
(2) $(i,j) \in P \implies$

$$\begin{cases} (i,j-1) \in P & \text{if } r_{i,j-1} = r_{ij} \\ (i+1,j) \in P & \text{if } r_{i,j-1} < r_{ij} \ \& \ (i+1,j) \in \lambda \\ \text{stop} & \text{otherwise} \end{cases}$$

__Example__

$$R = \begin{array}{cccc} 1 & 2 & 2 & 2 \\ 3 & 3 & 3 & \\ 3 & & & \end{array}$$

Suppose that p starts in row r & ends in column c



HG : $\begin{Bmatrix} \text{RPP} \\ R \end{Bmatrix} \to \begin{Bmatrix} \text{matrix} \\ A \end{Bmatrix}$

(1) Start with zero matrix A

(2) If RPP R has all zero entries then STOP.

(3) Otherwise find the ribbon path p in RPP.

r - the first row of p

c - the last column of p.

(4) subract 1's from all boxes in p in RPP

(5) add 1 to a_{rc}

(6) GO TO step (2)

Example $\varkappa = (4,3,1)$

$$R = \begin{array}{cccc} 1 & 2 & 2 & 2 \\ 3 & 3 & 3 & \\ 3 & & & \end{array} \quad h_{11}$$

\# 1's removed from RPP R is the hook length h_{11}

$$\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & \\ 2 & & & \end{array} \; h_{11} \;\rightsquigarrow\; \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & \\ 1 & & & \end{array} \; h_{23}$$

$$\rightsquigarrow\; \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 1 & 2 & 2 & \\ 1 & & & \end{array} \; h_{22} \;\rightsquigarrow\; \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & \\ 1 & & & \end{array} \; h_{21}$$

$$\rightsquigarrow\; \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \\ 0 & & & \end{array} \quad \text{STOP} \qquad A = \begin{array}{cccc} 2 & 0 & 0 & 0 \\ 1 & 1 & 1 & \\ 0 & & & \end{array}$$

HG: $\begin{array}{cccc} 1 & 2 & 2 & 2 \\ 3 & 3 & 3 & \\ 3 & & & \end{array} \rightsquigarrow \begin{array}{cccc} 2 & 0 & 0 & 0 \\ 1 & 1 & 1 & \\ 0 & & & \end{array}$

RPP R   matrix A

Clearly, $\sum r_{ij} = \sum h_{ij}\, a_{ij}$

Theorem. This construction gives a bijection between RPP's & matrices with needed properties.

Proof. Need to construct the inverse map

$$\underset{\text{matrix}}{A} \longmapsto \underset{\text{RPP}}{R}$$

(1) Start with RPP with all 0's.

(2) if $A = (0)$, then Stop.

(3) O.w. read the entries of A by rows from left to right from bottom to top

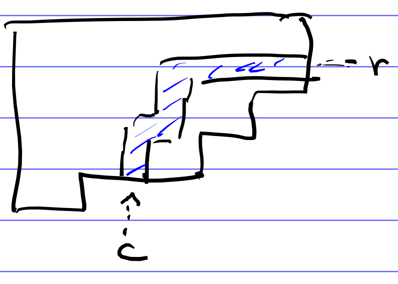
(Lemma: This is exactly opposite to the order in which 1's are added to A in the map $R \mapsto A$.)

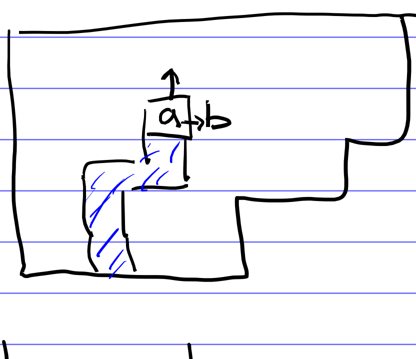
until we find a non-zero entry $a_{rc} > 0$ in A

(4) Find ribbon path P in RPP (by reversing the const. $R \rightsquigarrow A$)

- start at the bottom of column c.
- at each step

$\begin{cases} \text{go right if } a = b \\ \text{otherwise go up} \end{cases}$





- stop when you arrive to the rightmost box of row r.

(5) Add 1's to boxes of P in RPP.

Subtract 1 from a_{rc}.

(6) GO TO Step (2).

$\square$

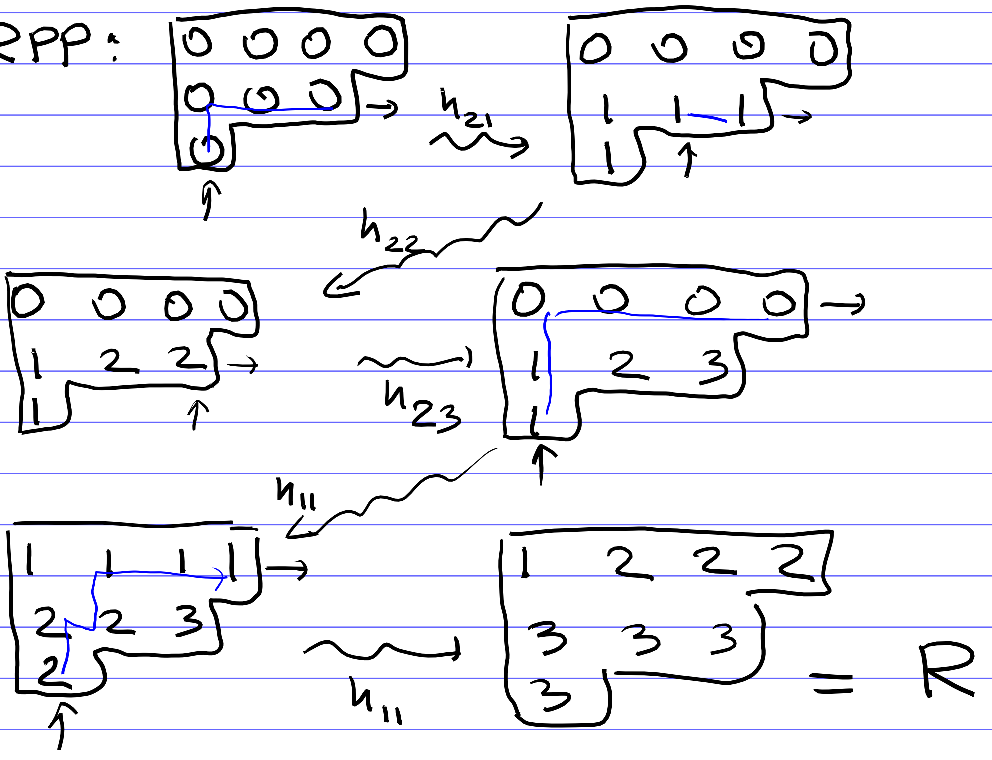
__Example__ $A = \begin{array}{|cccc|} \hline 2 & 0 & 0 & 0 \\ 1 & 1 & 1 & \\ 0 & & & \\ \hline \end{array}$

read the entries by rows left-to-right bottom-to-top

The sequence of ribbons that we will add to RPP:

$$h_{21}, \ h_{22}, \ h_{23}, \ h_{11}, \ h_{11}$$

RPP:



We've got 2 maps $\varphi^{RSK}_{\varkappa}$ & $\varphi^{HG}_{\varkappa}$ with exactly the same source & target sets and with the same properties.

Q. Are these maps the same?

__Problem.__

Prove or disprove:

$$\varphi^{RSK}_{\varkappa} \overset{?}{=} \varphi^{HG}_{\varkappa}$$

Theorem (R. Stanley)

$$s_\lambda(1, q, q^2, q^3, \dots)$$

sym. funct. in infinitely many variables

$$= q^{n(\lambda)} \prod_{(i,j)\in\lambda} \frac{1}{1-q^{h_{ij}}},$$

hook lengths

where $n(\lambda) := \sum_{i\geq 1} (i-1)\lambda_i$

Example. $s_{(3,2)}(1, q, q^2, \dots) =$

$$= \frac{q^2}{(1-q^4)(1-q^3)(1-q)(1-q^2)(1-q)}.$$

Proof We can use either map φ_λ^{RSK} or φ_λ^{HG}

We'll now use λ instead of $\mathcal{D}$.

$$\left\{\begin{matrix}\text{matrices}\\ A \text{ of}\\ \text{shape } \lambda\end{matrix}\right\} \underset{\varphi_\lambda^{HG}}{\overset{\varphi_\lambda^{RSK}}{\rightrightarrows}} \left\{\begin{matrix}\text{RPP's}\\ R \text{ of}\\ \text{shape } \lambda\end{matrix}\right\}$$

$$\sum h_{ij} a_{ij} = \sum r_{ij}$$

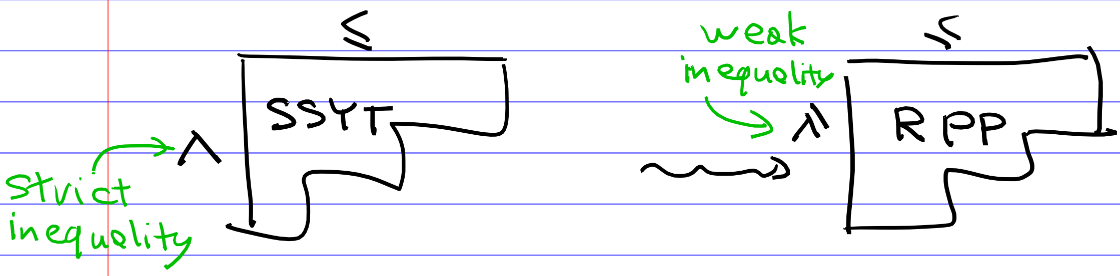
$$\Rightarrow \sum_{\substack{\text{all non-negative}\\ \text{integer matrices}\\ \text{of shape } \lambda}} q^{\sum h_{ij} a_{ij}} = \sum_{\substack{\text{all non-negative}\\ \text{integer RPPs}\\ \text{of shape } \lambda}} q^{\sum r_{ij}}$$

$$\| $$

$$\prod_{(i,j)\in\lambda} \frac{1}{1-q^{h_{ij}}}$$

$$s_\lambda(1, q, q^2, \dots) = \sum_{\substack{\text{SSYT's}\\ \text{of shape } \lambda\\ \text{filled with}\\ 0,1,2,3,\dots}} q^{\sum \text{entries}}$$

From SSYT's to RPP's



Subtract $(i-1)$ from all entries in row i, for $i = 1, 2, \ldots$

$$\begin{array}{cccccc} 0 & 0 & 0 & 1 & 1 & 4 \\ 1 & 1 & 2 & 2 & 5 & \\ 3 & 3 & 3 & 4 & & \\ 4 & 4 & 5 & & & \end{array} \quad \rightsquigarrow \quad \begin{array}{cccccc} 0 & 0 & 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & 1 & 4 & \\ 1 & 1 & 1 & 2 & & \\ 1 & 1 & 2 & & & \end{array}$$

T SSYT $\qquad$ R RPP

Clearly, $\sum\{\text{entries of } T\}$

$= n(\lambda) + \sum\{\text{entries of } R\}$.

So $s_\lambda(1, q, q^2, \ldots) =$

$$= q^{n(\lambda)} \cdot \sum_{\substack{\text{RPP's} \\ \text{of shape } \lambda}} q^{\sum r_{ij}}$$

$$= q^{n(\lambda)} \prod_{(i,j)\in\lambda} \frac{1}{1-q^{h_{ij}}}. \qquad \square$$

Hook length formula

(Frame–Robinson–Thrall)

$$\underset{\substack{\#\text{ SYT's of} \\ \text{shape } \lambda \vdash n}}{\overset{\Vert}{f_\lambda}} = \frac{n!}{\prod\limits_{(i,j)\in\lambda} h_{ij}}$$

Example. $\lambda = \begin{array}{|c|c|c|} \hline 4 & 3 & 1 \\ \hline 2 & 1 \\ \cline{1-2} \end{array}$ hooklengths

$$f_{\lambda} = \frac{5!}{4\cdot 3\cdot 2\cdot 1\cdot 1} = 5$$

5 SYT's: $\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} \end{array}$ $\begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2} \end{array}$

$\begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2} \end{array}$ $\begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2} \end{array}$ $\begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2} \end{array}$

Proof. Let's use

φ_λ^{RSK} (not φ_λ^{HG})

because we know that φ_λ^{RSK} extends to a volume preserving map between real matrices & RPP's with real entries.

(One can also use the Hillman–Grassl correspondence to prove the hook length formula.)

Define 2 convex polytopes in $\mathbb{R}^n$, $n = |\lambda|$.

P_{RPP} = the polytope of all RPP's $R = (r_{ij})$ of shape λ with real entries $r_{ij} \geq 0$ with $\sum r_{ij} \leq 1$

here we can take any other number

P_{Mat} = the polytope of real nonnegative matrices $A = (a_{ij})$ of shape λ with

$$\sum h_{ij} a_{ij} \leq 1$$

Ex $\lambda = (3,2)$

$$\begin{array}{|c|c|c|} \hline x & y & z \\ \hline u & v & \\ \cline{1-2} \end{array}$$

$$P_{RPP} := \left\{ (x,y,z,u,v) \in \mathbb{R}^5 \;\middle|\; \begin{array}{c} \begin{array}{ccccccc} 0 \leq & x & \leq & y & \leq & z \\ & \rotatebox[origin=c]{-90}{$\leq$} & & \rotatebox[origin=c]{-90}{$\leq$} & & \\ & u & \leq & v & & \end{array} \\ x+y+z+u+v \leq 1 \end{array} \right\}$$

$$P_{Mat} := \left\{ (x,y,z,u,v) \in \mathbb{R}^5 \;\middle|\; \begin{array}{c} x, y, z, u, v \geq 0 \\ 4x + 3y + z + 2u + v \leq 1 \end{array} \right\}$$

φ_λ^{RSK} is a volume preserving bijection between P_{Mat} & P_{RPP}.

In particular, $\mathrm{Vol}\, P_{Mat} = \mathrm{Vol}\, P_{RPP}$

P_{RPP} can be subdivided into f_λ parts given by picking a total ordering of entries in RPPs.

Ex

$$\begin{array}{ccc} x & \le y & \le z \\ \rotatebox{90}{$\le$} & \rotatebox{90}{$\le$} & \\ u & \le v & \end{array}$$

$$P_{RPP} = \left\{ \begin{array}{c} 0 \le x \le y \le z \le u \le v, \\ x + y + \ldots + v \le 1 \end{array} \right\} \cup$$

$$\cup \left\{ \begin{array}{c} 0 \le x \le y \le u \le z \le v, \\ x + y + \ldots + v \le 1 \end{array} \right\} \cup \ldots$$

(5 possible orderings of entries)

So $\mathrm{Vol}\, P_{RPP} =$

$$f_\lambda \cdot \mathrm{Vol} \underbrace{\left\{ \begin{array}{c} 0 \le x_1 \le \ldots \le x_n \\ \sum x_i \le 1 \end{array} \right\}}_{\text{this is an } n\text{-dim simplex}}$$

Exercise Calculate Vol of this simplex (We don't really need it for the proof)

On the other hand,

$$\mathrm{Vol}(P_{Mat}) = \mathrm{Vol}\left\{ \begin{array}{c} a_{ij} \geq 0 \\ \sum h_{ij} a_{ij} \leq 1 \end{array} \right\}$$

$$= \frac{\mathrm{Vol}\left\{ \begin{array}{c} \tilde{a}_{ij} \geq 0 \\ \sum \tilde{a}_{ij} \leq 1 \end{array} \right\}}{\prod_{(i,j)\in\lambda} h_{ij}}$$

we rescale the variables a_{ij} as $\tilde{a}_{ij} = h_{ij} a_{ij}$

$n!$ possible orderings of $\tilde{a}_{ij}$

$$= \frac{n!}{\prod h_{ij}} \mathrm{Vol}\left\{ \begin{array}{c} 0 \leq x_1 \leq \ldots \leq x_n, \\ \sum x_i \leq 1 \end{array} \right\}$$

the same n-dim simplex

So

$$f_\lambda = \frac{n!}{\prod h_{ij}},$$

as needed. $\square$

More on volumes of polytopes...

Gelfand–Tsetlin polytope $GT_\lambda \subseteq \mathbb{R}^{\binom{n}{2}}$

$\lambda = (\lambda_1, \dots, \lambda_n)$

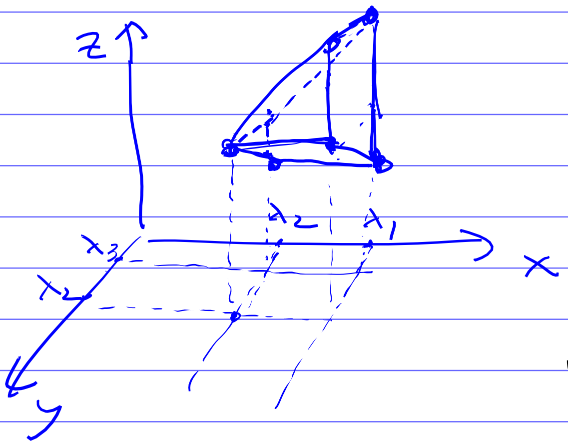
– the polytope of all GT-patterns with real entries with top row λ

Ex. $n = 3$:

$$\begin{array}{ccccccccc} \lambda_1 & & & & \lambda_2 & & & & \lambda_3 \\ & \geq & & \geq & & \geq & & \geq & \\ & & x & & & & y & & \\ & & & \geq & & \geq & & & \\ & & & & z & & & & \end{array}$$

$GT(\lambda_1, \lambda_2, \lambda_3) :=$

$= \{ (x, y, z) \in \mathbb{R}^3 \mid$ satisfying these inequalities $\}$



$\mathrm{Vol}\; GT(\lambda) = ?$

$\mathrm{Vol}\; GT(\lambda_1, \lambda_2, \lambda_3) =$

$$= \int_{\lambda_3}^{\lambda_2} \int_{\lambda_2}^{\lambda_1} (y - x)\, dx\, dy$$

$$= \int_{\lambda_3}^{\lambda_2} \left(\left(xy - \frac{x^2}{2} \right) \Big|_{x=\lambda_2}^{x=\lambda_1} \right) dy$$

$$= \frac{(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)(\lambda_1 - \lambda_3)}{2}$$

__Theorem__ $\operatorname{Vol} GT(\lambda_1, \ldots, \lambda_n)$

$$= \frac{1}{1!\,2!\,\ldots(n-1)!} \prod_{1 \le i < j \le n} (\lambda_i - \lambda_j).$$

We deduced the hook length formula by calculating volumes of polytopes. Let's go in the opposite direction...

__Stanley's__ __Hook - content Formula__

$$S_\lambda(1, q, q^2, \ldots, q^{n-1}) =$$

content

$$= q^{n(\lambda)} \prod_{(i,j)\in\lambda} \frac{1-q^{n+c_{ij}}}{1-q^{h_{ij}}}$$

hook length

content of box (i,j)

$c_{ij} = j - i$

__Ex__. $\lambda = (3,2)$, $S_{(3,2)}(1, q, q^2, \ldots, q^{n-1})$

$$= q^2 \frac{(1-q^n)^2(1-q^{n+1})(1-q^{n+2})(1-q^{n-1})}{(1-q^4)(1-q^3)(1-q)(1-q^2)(1-q)}$$

Setting $q \to 1$.

We get

$$S_\lambda(\underbrace{1,\dots,1}_{n}) = \#\left\{\begin{array}{l}\text{SSYT of shape } \lambda \\ \text{with entries} \\ \in \{1,2,\dots,n\}\end{array}\right\}$$

$$= \boxed{\prod_{(i,j)\in\lambda} \frac{n+c_{ij}}{h_{ij}}}$$

Also called hook-content formula

Actually, there is another formula for $S_\lambda(\underbrace{1,\dots,1}_{n})$ due to Weyl.

Weyl's dimension Formula
(of type A)

$\lambda = (\lambda_1, \dots, \lambda_n)$ partition

$$S_\lambda(\underbrace{1,\dots,1}_{n}) = \#\left\{\begin{array}{l}\text{SSYT's of} \\ \text{shape } \lambda \text{ with} \\ \text{entries} \in \{1,\dots,n\}\end{array}\right\}$$

$$= \#\left\{\begin{array}{l}\text{integer GT-patterns} \\ \text{with top row } \lambda\end{array}\right\}$$

$$= \boxed{\prod_{1\le i<j\le n} \frac{\lambda_i - \lambda_j + j - i}{j-i}}$$

Exercise. Prove combinatorially that the hook-content formula for $s_\lambda(\underbrace{1,\dots,1}_{n})$ is equivalent to Weyl's dim. formula

$$\prod_{a\in\lambda}\frac{n+c_a}{h_a}\overset{?}{=}\prod_{1\le i<j\le n}\frac{\lambda_i-\lambda_j+j-i}{j-i}.$$

Example. $n=3$, $\lambda=(3,2,0)$

$s_{\text{⊞}}(1,1,1)=$

$$=\frac{3\cdot4\cdot5\cdot2\cdot3}{4\cdot3\cdot1\cdot2\cdot1}=\frac{2\cdot3\cdot5}{1\cdot1\cdot2}.$$

hook-content formula

Weyl's dimension formula.

Back to GT-polytope...

Weyl's dimension formula gives $s_\lambda(\underbrace{1\dots1}_{n})=\#$ integer lattice points in $GT(\lambda)$.

To get volume we need to dilate & take the limit.

Lemma For any polytope $P\subset\mathbb{R}^N$,

$$\mathrm{Vol}(P)=\lim_{t\to\infty}\frac{\#(tP\cap\mathbb{Z}^N)}{t^N}.$$

Actually, this is basically the definition of Volume.

For GT-polytope $\subset \mathbb{R}^{\binom{n}{2}}$

we get

$$\mathrm{Vol}\, GT(\lambda_1, \ldots, \lambda_n) =$$

$$= \lim_{t \to \infty} \frac{\prod_{1 \le i < j \le n} \frac{t(\lambda_i - \lambda_j) + j - i}{j - i}}{t^{\binom{n}{2}}}$$

$$= \prod_{1 \le i < j \le n} \frac{\lambda_i - \lambda_j}{j - i} \qquad \square$$