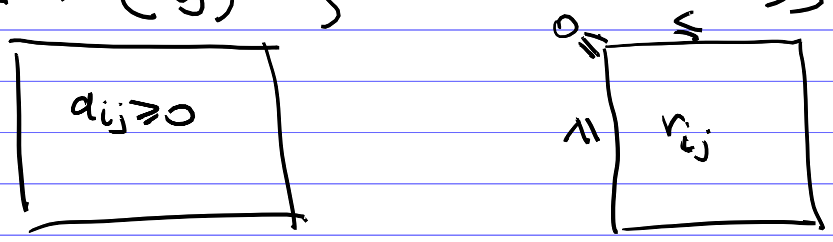
18.217 Lecture 10 09/25/2020

RSK: $\left\{\begin{array}{c} n\times n \text{ matrices} \\ A=(a_{ij}) \end{array}\right\} \longleftrightarrow \left\{\begin{array}{c} n\times n \\ \text{RPP's} \end{array}\right\}$



$n\times n$ RPP represent 2 Gelfand-Tsetlin patterns "glued along λ"

$\Updownarrow$

2 SSYT's P, Q of shape 2.

This way to view RSK in terms "glued Gelfand-Tsetlin patterns" was introduced by **Berenstein & Kirillov**

Ex $n=4$

RPP: $$\rotatebox{90}{$\leq$}\ \overset{\leq}{\begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 1 & 2 & 4 \\ 2 & 3 & 3 & 5 \\ 3 & 3 & 4 & 5 \end{bmatrix}}$$

$\lambda=(5,3,1,0)$

2 GT patterns with top row λ

$$\begin{array}{ccccccc} 5 & & 3 & & 1 & & 0 \\ & 4 & & 3 & & 1 & \\ & & 3 & & 2 & & \\ & & & 3 & & & \end{array} \qquad \begin{array}{ccccccc} 5 & & 3 & & 1 & & 0 \\ & 5 & & 2 & & 1 & \\ & & 4 & & 1 & & \\ & & & 2 & & & \end{array}$$

SSYT's

$$P=\begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 3 & 4 \\ \hline 2 & 2 & 3 \\ \cline{1-3} 3 \\ \cline{1-1} \end{array} \qquad Q=\begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 3 \\ \hline 2 & 3 & 4 \\ \cline{1-3} 3 \\ \cline{1-1} \end{array}$$

$\text{weight}(P)=(3,2,3,1)$ $\text{weight}(Q)=(2,3,3,1)$

Properties of RSK :

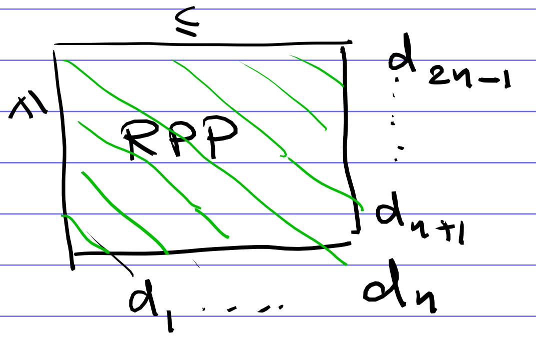
- Column sums of A ⟷ weight of P
- Row sums of A ⟷ weight of Q

---

weights of P, Q ⟺ differences of diagonal sums of RPP.

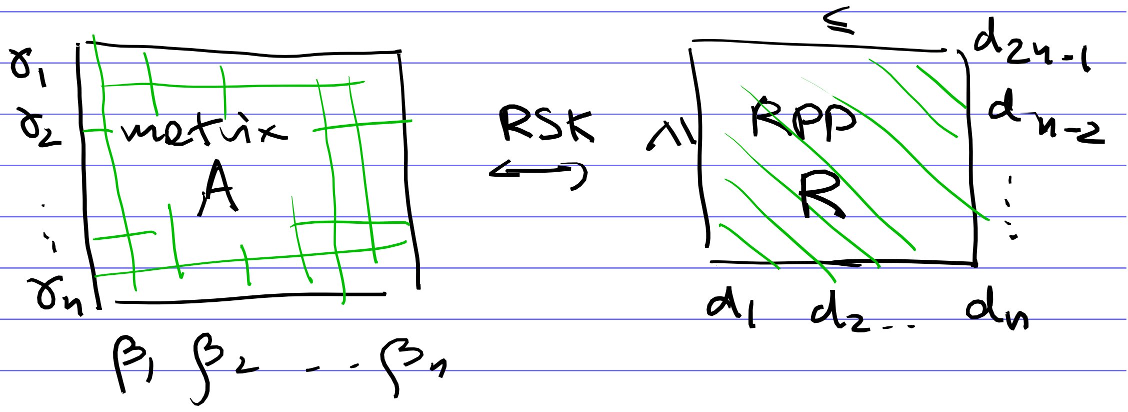
$wt(P) = (\beta_1, \ldots, \beta_n)$, $wt(Q) = (\gamma_1, \ldots, \gamma_n)$

$$\beta_1 + \ldots + \beta_n = \gamma_1 + \ldots + \gamma_n$$

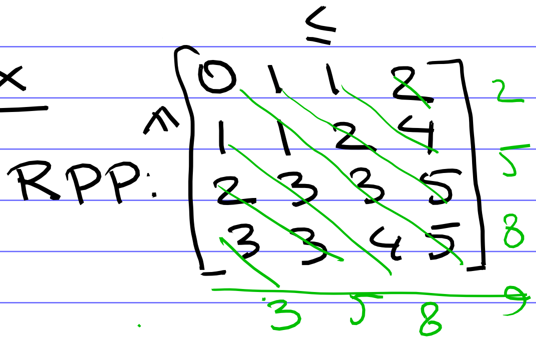


d_i - diagonal sums RPP corresp. to (P,Q)

$\beta_1 = d_1$	$\gamma_1 = d_{2n-1}$
$\beta_2 = d_2 - d_1$	$\gamma_2 = d_{2n-2} - d_{2n-1}$
$\beta_3 = d_3 - d_2$	$\gamma_3 = d_{2n-3} - d_{2n-2}$
$\ldots$	$\ldots$
$\beta_n = d_n - d_{n-1}$	$\gamma_n = d_n - d_{n+1}$



Ex



RPP: $\begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 1 & 2 & 4 \\ 2 & 3 & 3 & 5 \\ 3 & 3 & 4 & 5 \end{bmatrix}$

$wt(P) = (\beta_1, \ldots, \beta_4)$
$= (3, 2, 3, 1)$

$wt(Q) = (\gamma_1, \ldots, \gamma_4)$
$= (2, 3, 3, 1)$

$(d_1, \ldots, d_{2n-1}) = (3, 5, 8, 9, 8, 5, 2)$

RSK can be generalized in two directions:

- The shape of matrix A & RPP R can be any Young diagram λ (not necessarily $n \times n$ square)

- Entries of A and R can be any nonnegative real numbers (not necessarily integers)

Ex.

$$\begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline e & \\ \cline{1-1}\end{array} \longleftrightarrow \begin{array}{cc} \overset{0}{\rotatebox{90}{$\ge$}} & \le \\ \end{array} \begin{array}{c} \\ \ge \\ \\ \end{array} \begin{array}{|c|c|}\hline \min(b,c) & a+b \\ \hline a+c & a+d+\max(b,c) \\ \hline a+c+e & \\ \cline{1-1}\end{array}$$

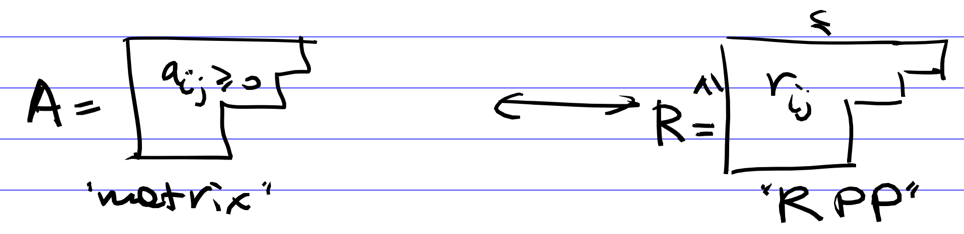
$a, b, c, d, e \geq 0$ real numbers

"RPP" with nonnegative real entries

Fix a Young diagram $\varkappa$

<u>Theorem</u>. $\exists$ bijection $\varphi_\varkappa$

$$\left\{\begin{matrix}\text{non-negative}\\ \text{real matrices}\\ \text{of shape } \varkappa\end{matrix}\right\} \overset{\varphi_\varkappa}{\longleftrightarrow} \left\{\begin{matrix}\text{non-negative}\\ \text{real RPPs}\\ \text{of shape } \varkappa\end{matrix}\right\}$$



such that

- $\varphi_\varkappa$ is a <u>piecewise-linear</u> <u>volume preserving</u> map.

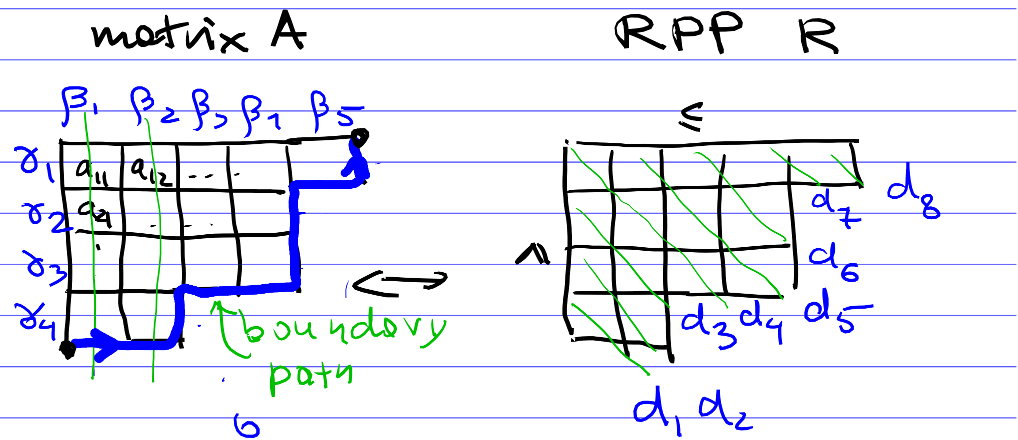
- $\varphi_\varkappa$ induces a bijection between <u>integer</u> matrices & <u>integer</u> RPPs

- Column sums & row sums of A are linearly related to diagonal sums of R, as follows:

Consider the <u>boundary path</u> from lower left to upper right corners of $\varkappa$

$$d_i - d_{i-1} = \begin{cases} \begin{pmatrix}\text{column sum}\\ \text{above the step}\end{pmatrix} & \begin{matrix}\text{if the } i^{th}\\ \text{step in the}\\ \text{boundary path}\\ \text{is horizontal}\end{matrix} \\ -\begin{pmatrix}\text{row sum to}\\ \text{the left of the step}\end{pmatrix} & \begin{matrix}\text{if —"—}\\ \text{is vertical}\end{matrix}\end{cases}$$

<u>Ex</u>:



$$\underset{\overset{\parallel}{0}}{}$$

$d_1 - \overset{\overset{0}{\parallel}}{d_0} = \beta_1$

$d_2 - d_1 = \beta_2$

$d_3 - d_2 = -\gamma_4$

$d_4 - d_3 = \beta_3$

$d_5 - d_4 = \beta_4$

$d_6 - d_5 = -\gamma_3$

$d_7 - d_6 = -\gamma_2$

$d_8 - d_7 = \beta_5$

$\underset{\overset{\parallel}{0}}{d_9} - d_8 = -\gamma_1$

From different points of view bijection φ_x generalizes:

- RSK
- Fomin's growth diagrams
- Oscillating tableaux
- Hillman - Grassl correspondence

This approach was introduced by Berenstein - Kirillov.

It is related to

- tropical / geometric RSK
- octahedron recurrence
- cluster algebras.

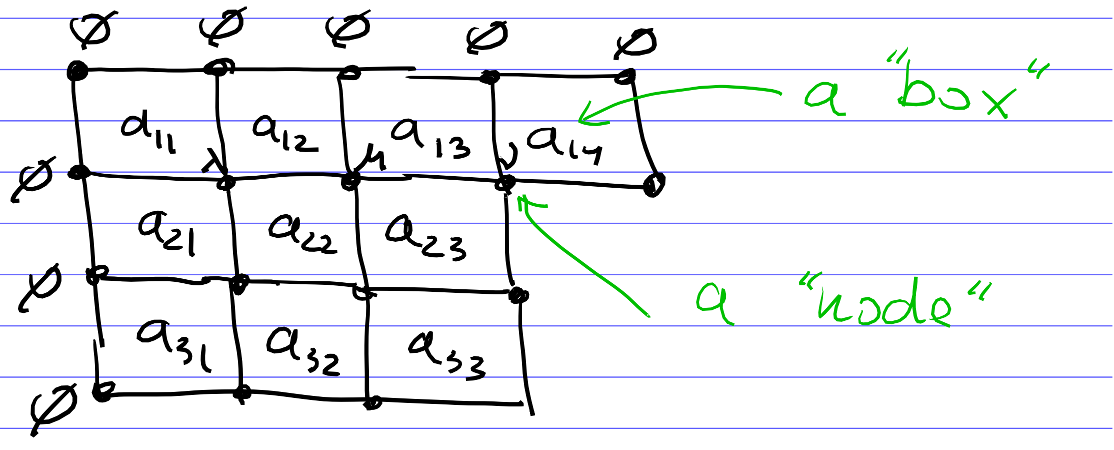
From some point of view all above notions are the same thing.

Construction of $\varphi_{\varkappa}$:

A version of growth diagrams for semi-standard case

we'll the growth diagr. from prev. lect upside down

Semi-standard growth diagrams of shape $\varkappa$



- boxes of $\varkappa$ are filled with $a_{ij} \geq 0$
- nodes of $\varkappa$ are filled with Young diagrams λ
- empty Young diag. in all nodes in 1st row and 1st column

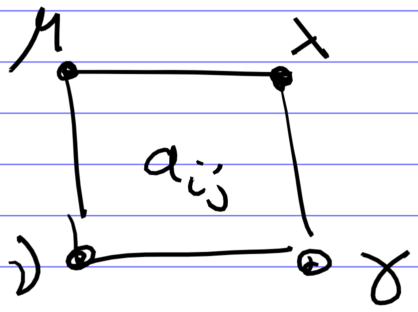
$\forall$ two adjacent nodes

or

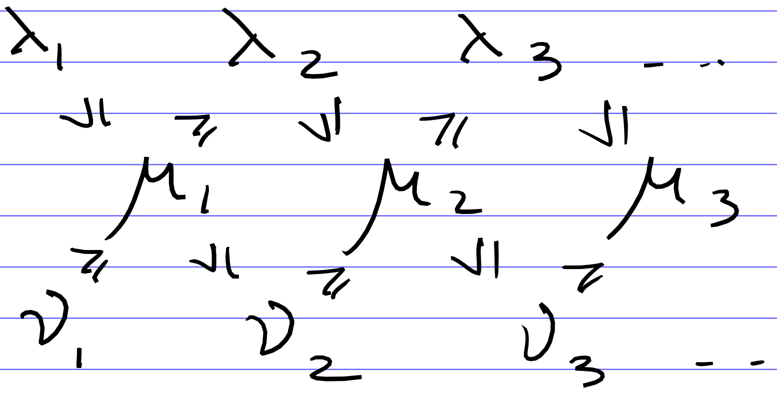
$\lambda \supseteq \mu$ and λ/μ is a horizontal strip, i.e. the parts of λ & μ are interlaced:

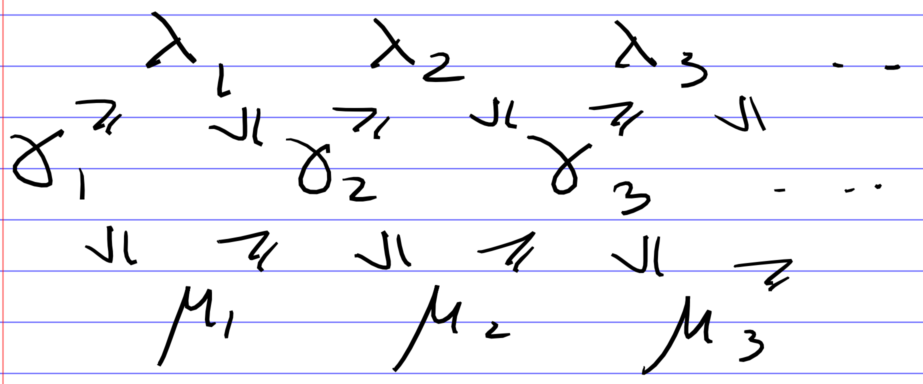
$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3 \geq \mu_3 \geq \lambda_4 \geq \mu_4 \ \ldots$$

There is only 1 local rule:



For each box of $\mathcal{D}$

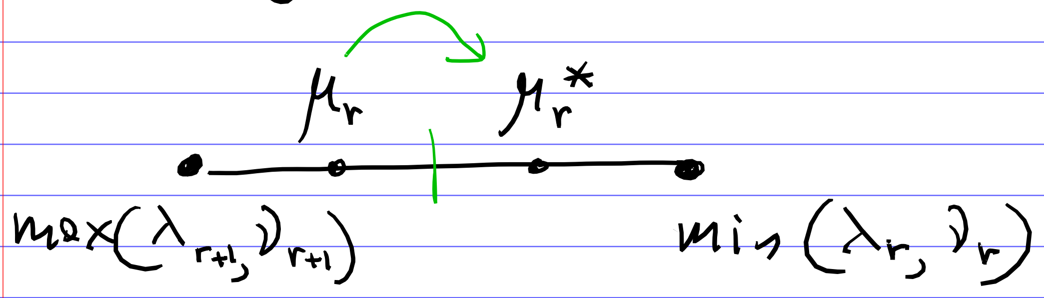




or

$$\mu_r, \gamma_{r+1} \in [\min(\lambda_r, \nu_r), \max(\lambda_{r+1}, \nu_{r+1})]$$

Toggle Operations



reflect μ_r with respect to the mid-point of the interval

$$\boxed{\mu_r^* + \mu_r = \min(\lambda_r, \nu_r) + \max(\lambda_{r+1}, \nu_{r+1})}$$

Local Rule:

$$\begin{tikzcd}
\mu \arrow[r, dash] \arrow[d, dash] \arrow[dr, phantom, "a_{ij}"] & \lambda \arrow[d, dash] \\
\nu \arrow[r, dash] & \gamma
\end{tikzcd}$$

γ is determined by λ, μ, ν and a_{ij} as follows

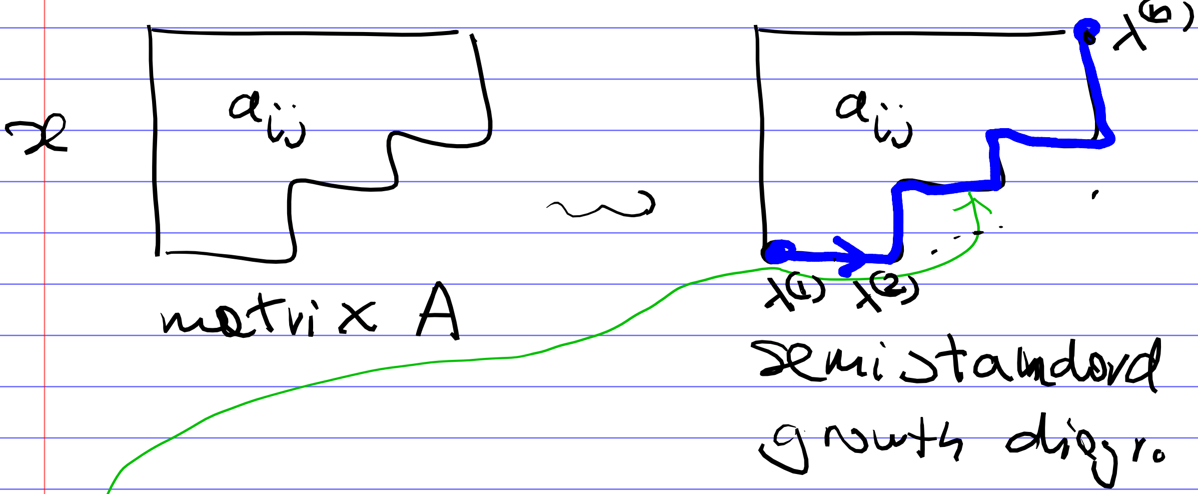
$$\begin{cases} \gamma_{r+1} = \min(\lambda_r, \nu_r) + \max(\lambda_{r+1}, \nu_{r+1}) \\ \qquad\qquad - \mu_r, \quad \text{for } r \geq 1 \\ \gamma_1 = \max(\lambda_1, \nu_1) + a_{ij} \end{cases}$$

γ_{r+1} is the toggle of μ_r

Lemma $\forall$ matrix $A = (a_{ij})$, $a_{ij} \geq 0$ of shape $\varkappa$, there exists a unique semi-standard growth diagram.

Proof Easy. Basically, the same as for usual growth diagrams. $\square$

Construction of $\varphi_\varkappa$



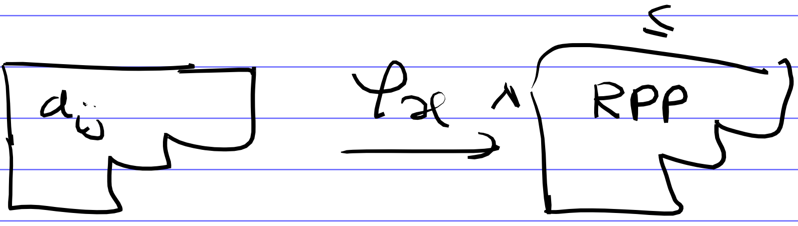
For each node on the boundary of $\varkappa$ we have a Young diagr.

We get a sequence of Young diar $\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}, \ldots$

s.t. parts of $\lambda^{(i)}$ are interlaced with parts of $\lambda^{(i+1)}$

Put the parts of $\lambda^{(i)}$ into the ith diagonal of RPP of shape $\varkappa$.

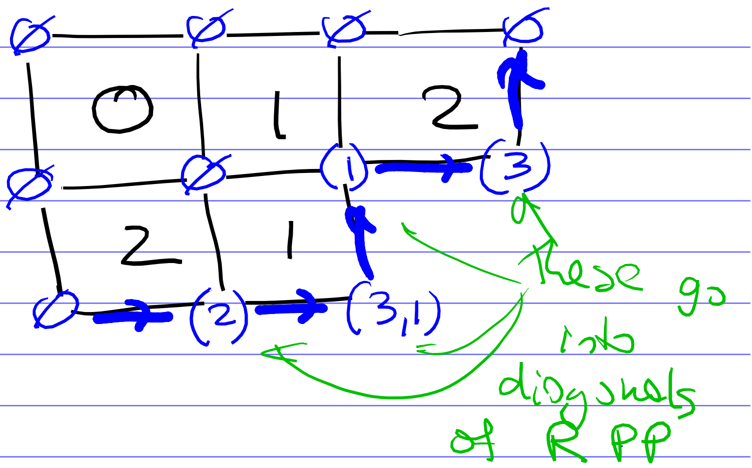
This gives the corresp.

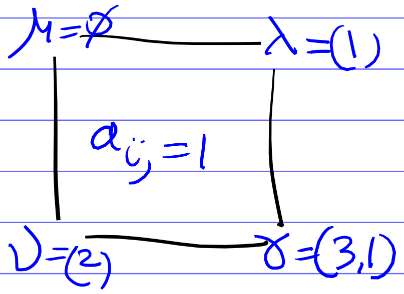


Example

$$A = \begin{array}{|c|c|c|}\hline 0 & 1 & 2 \\ \hline 2 & 1 \\ \cline{1-2}\end{array}$$

growth diagram





$$\begin{array}{lccccc}
\lambda : & 1 & & 0 & & 0\;.. \\
 & \geq & \geq & \geq & \geq & \geq \\
\mu : & & 0 & & 0 & & 0 \\
 & \geq & \geq & \geq & \geq & \geq \\
\nu : & 2 & & 0 & & 0
\end{array}$$

$$\gamma = (3, \quad 1)$$

$\max(\lambda_1, \nu_1) + a_{ij}$ ↗ 3

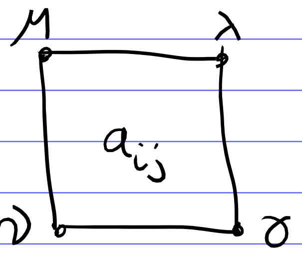
↑ toggle of μ_1

$$\text{RPP} = \begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & 3 \\ \cline{1-2}\end{array}$$

$$\underset{A}{\begin{array}{|c|c|c|}\hline 0 & 1 & 2 \\ \hline 2 & 1 \\ \cline{1-2}\end{array}} \xrightarrow{\varphi_{\boxplus}} \underset{R}{\overset{\leq}{\begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & 3 \\ \cline{1-2}\end{array}}}$$

It is easy to see that this construction is reversable

the local rule is reversable:



$\lambda, \mu, \nu, a_{ij} \rightsquigarrow \gamma$

$$\begin{cases} \gamma_{r+1} = \text{toggle of } \mu_r, \ r \geq 1 \\ \gamma_1 = a_{ij} + \max(\lambda_1, \nu_1) \end{cases}$$

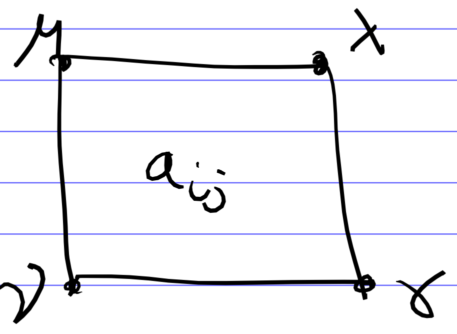
$\lambda, \gamma, \gamma \rightsquigarrow \mu, a_{ij}$

$$\begin{cases} \mu_r = \text{toggle of } \gamma_{r+1}, \ r \geq 1 \\ a_{ij} = \gamma_1 - \max(\lambda_1, \nu_1) \end{cases}$$

Also not hard to see row/col. sums of matrix A are related to diag. sums of RPP R as needed

Also follows by ind. from local rule:

Lemma



$$|\gamma| = |\lambda| - |\mu| + |\nu| + a_{ij}.$$

this is diag. sum of RPP

Proof $|\gamma| =$

$= a_{ij} + \max(\lambda_1, \nu_1)$ ← γ_1

$+ \min(\lambda_1, \nu_1) + \max(\lambda_2, \nu_2) - \mu_1$ ← γ_2

$+ \min(\lambda_2, \nu_2) + \max(\lambda_3, \nu_3) - \mu_2$ ← γ_3

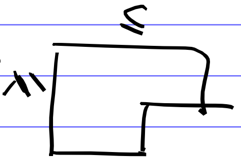
$+ \ldots$

$= a_{ij} + \sum \lambda_i + \sum \nu_i - \sum \mu_i$

$\square$

__Theorem__ The above construction (via toggles & semi-standard growth diagrams) give a well defined bijection

$$\varphi_{x} : \left\{ \boxed{a_{ij} \geq 0} \right\} \xrightarrow{\sim} \left\{ \quad \right\}$$



RPP's

That has all the properties that we mentioned above, and also

- symmetry under (transposition of matrices, conjugation of shapes

$$\varphi_{x} : A \longmapsto T$$

$$\varphi_{x'} : A^{T} \longmapsto T^{T}$$

and many other nice properties ....

Now let's prove

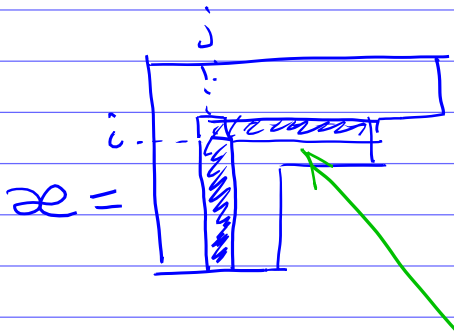
the hook-length formula...

Hooklength Formula

(Frame – Robinson – Thrall)

Let $\varkappa$ be a Young diagram with $|\varkappa| = N$ boxes.

$$\#\left\{\begin{matrix}\text{SYT's}\\ \text{of shape}\\ \varkappa\end{matrix}\right\} = \frac{N!}{\prod\limits_{(i,j)\in\varkappa} h_{ij}}$$

$\varkappa =$ 

h_{ij} = "hooklength"

= the number of boxes in the hook at box (i,j) of $\varkappa$.

Ex. $\varkappa = \begin{array}{|c|c|c|}\hline 4 & 3 & 1 \\ \hline 2 & 1 & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array}$ hooklengths

SYT's of this shape

$$= \frac{5!}{4\cdot 3\cdot 2\cdot 1\cdot 1} = 5$$

5 SYT's:

$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array}$ $\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array}$ $\begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array}$

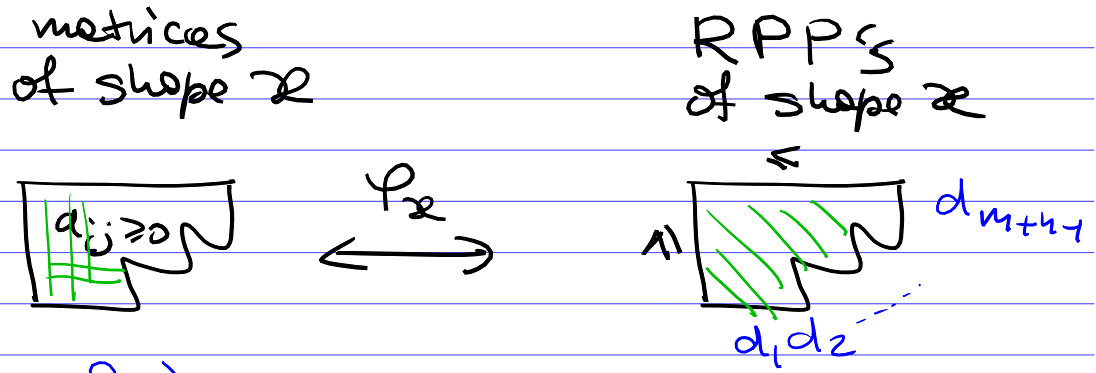
$\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array}$ $\begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array}$

At first glance there is no relationship between RSK & hook length formula...

How do we get hook lengths in RSK?

But actually it gives the "easiest" proof of the hook length formula...

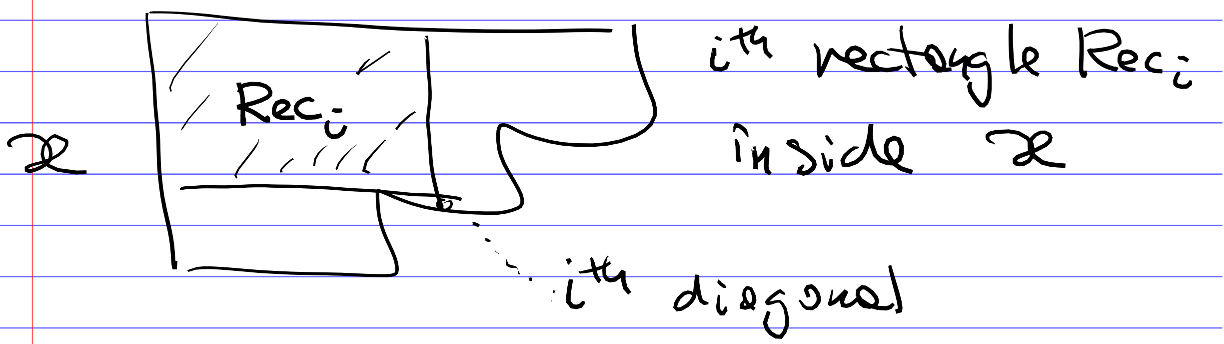
Let's explain this...



$(\beta_1, \ldots \beta_n)$
$(\gamma_1, \ldots, \gamma_m)$
column & row sums of matrix A

$(d_1, \ldots, d_{m+n-1})$
diagonal sums of RPP R

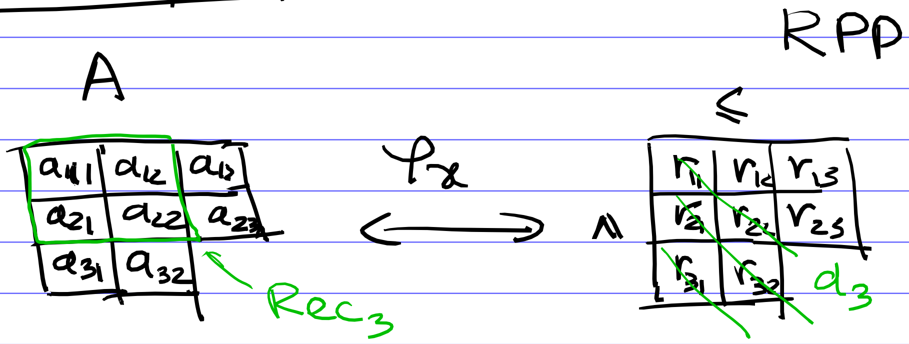
We can express the relationship between row/column sums of A and diagonal sums of R, as follows:



$$\boxed{d_i = \text{sum of entries of } A \text{ in } Rec_i}$$

ith diagonal sum of RPP

Example



$$a_{11} + a_{21} + a_{31} = d_1 = r_{31}$$

$$a_{11} + a_{12} + a_{21} + a_{22} + a_{31} + a_{32} = d_2 = r_{21} + r_{32}$$

$$a_{11} + a_{12} + a_{21} + a_{22} = d_3 = r_{11} + r_{22} + r_{33}$$

$$a_{11} + a_{12} + a_{13} + a_{21} + a_{22} + a_{23} = d_4 = r_{12} + r_{23}$$

$$a_{11} + a_{12} + a_{13} = d_5 = r_{13}$$

How about the sum of all entries in RPP?

In above example,

$$d_1 + d_2 + d_3 + d_4 = 5a_{11} + 4a_{12} + 2a_{13} + 4a_{21} + 3a_{22} + a_{23} + 2a_{31} + a_{32}$$

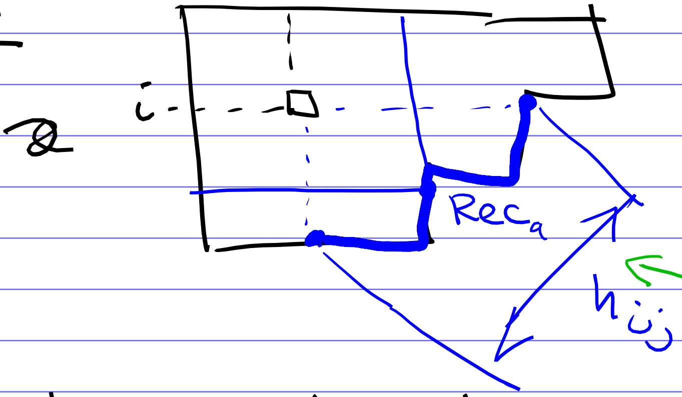
Lemma

$$\sum r_{ij} = \sum h_{ij} a_{ij}$$

sum of all entries of RPP $= \sum d_i$

sum of entries of matrix A weighted by the hook lengths h_{ij}

Proof



Each box (i,j) of a is contained in exactly h_{ij} Rectangles Rec_a. Namely, in all rectangles Rec_a for a's in this interval of size h_{ij}. □

Now let's calculate volumes of 2 polytopes in $\mathbb{R}^N$, $N = |\lambda|$.

P_{RPP} = the polytope of all RPPs $R = (r_{ij})$ of shape λ with real entries $r_{ij} \geq 0$ with $\sum r_{ij} \leq B$

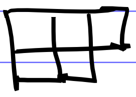
P_{Mat} = the polytope of all matrices $A = (a_{ij})$ of shape λ with

$$\sum h_{ij} a_{ij} \leq B$$

$$\varphi_\lambda : P_{Mot} \longleftrightarrow P_{RPP}$$

is a volume preserving map.

So $\mathrm{Vol}(P_{Mat}) = \mathrm{Vol}(P_{RPP})$

Ex. $\lambda =$ 

$$\begin{array}{|c|c|c|}\hline a & b & c \\ \hline d & e \\ \cline{1-2} \end{array}$$

matrix

$$\begin{array}{c} \qquad\qquad \leq \\ \underline{\wedge} \ \begin{array}{|c|c|c|}\hline x & y & z \\ \hline t & u \\ \cline{1-2} \end{array} \end{array}$$

RPP

$$P_{mot} = \left\{ \begin{array}{l} a, b, c, d, e \geq 0 \\ 4a + 3b + c + 2d + e \leq B \end{array} \right\}$$

$$P_{RPP} = \left\{ \begin{array}{ccccccc} 0 & \leq & x & \leq & y & \leq & z \\ & & \underline{\wedge} & & \underline{\wedge} & & \\ & & t & \leq & u & & \\ \multicolumn{7}{c}{x + y + z + t + u \leq B} \end{array} \right\}$$

P_{RPP} is subdivided into $\#SYT(\varkappa)$ parts of the same volume given by picking a total ordering of entries of an RPP

$$\mathrm{Vol}(P_{RPP}) = \#SYT(\varkappa) \cdot$$

a certain N-dim simplex

$$\mathrm{Vol}\left(\begin{Bmatrix} 0 \le x_1 \le \ldots \le x_N \\ \sum x_i \le B \end{Bmatrix}\right)$$

$$\mathrm{Vol}(P_{Mat}) = \mathrm{Vol}\left(\begin{Bmatrix} a_{ij} \ge 0 \\ \sum h_{ij} a_{ij} \le B \end{Bmatrix}\right)$$

$$\boxed{\tilde{a}_{ij} = h_{ij} a_{ij}}$$

$$= \frac{\mathrm{Vol}\begin{Bmatrix} \tilde{a}_{ij} \ge 0 \\ \sum \tilde{a}_{ij} \le B \end{Bmatrix}}{\prod_{(i,j)\in\varkappa} h_{ij}}$$

$$= \frac{N!}{\prod h_{ij}} \mathrm{Vol}\left(\begin{Bmatrix} 0 \le x_1 \le \ldots \le x_N, \\ \sum x_i \le B \end{Bmatrix}\right)$$

$$\mathrm{Vol}(P_{Mot}) = \mathrm{Vol}(P_{RPP})$$

$$\Big\downarrow$$

$$\frac{N!}{\prod h_{ij}} = \#\text{SYT's of shape } \lambda$$

Done. $\square$