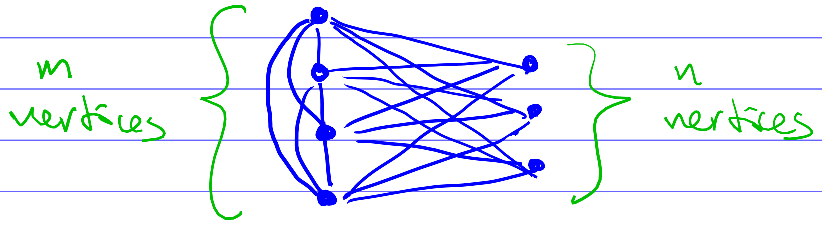
18.212 # Problem Set 3

due Friday, May 14, 2021

1. Find a formula for the number of spanning trees in the complete bipartite graph $K_{m,n}$, and prove it bijectively.

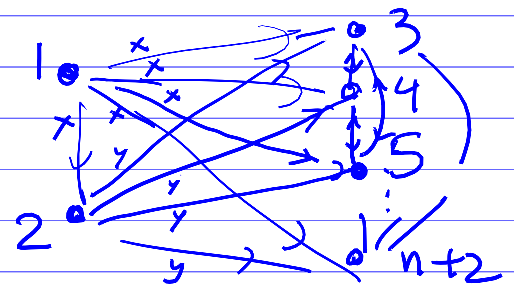
2. Give an explicit formula for the number of spanning trees in the complete tripartite graph $K_{m,n,k}$.

3. Give an explicit formula for the number of spanning trees in the graph G on $m+n$ vertices $1, 2, \ldots, m+n$ such that vertices i and j are connected by an edge if and only if at least one of i and j belongs to $\{1, 2, \ldots, m\}$.

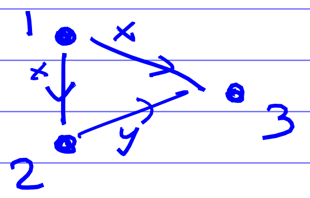
$G =$ 

(4) Consider the following weighted directed graph G on the $n+2$ vertices $1, 2, \dots, n+2$:

- G has directed edges $1 \to i$ of weight x for all $i \in \{2, \dots, n+2\}$. But G has no directed edge entering the vertex 1.
- G has directed edges $2 \to i$ of weight y for all $i \in \{3, 4, \dots, n+2\}$. But the only edge entering the vertex 2 is the edge $1 \to 2$
- Any other pair of vertices $i \neq j$, $i, j \in \{3, 4, \dots, n+2\}$ is connected by edge $i \to j$ of weight 1.

$G =$ 

(A) Find an explicit formula for the sum $\sum_T \text{weight}(T)$ over all arborescences T in G rooted at vertex 1.

For example, for $n=1$, the graph $G =$ 

has 2 abroresences rooted at 1, and the sum is $x^2 + xy = x(x+y)$.

(B) Deduce from (A) the following identity:

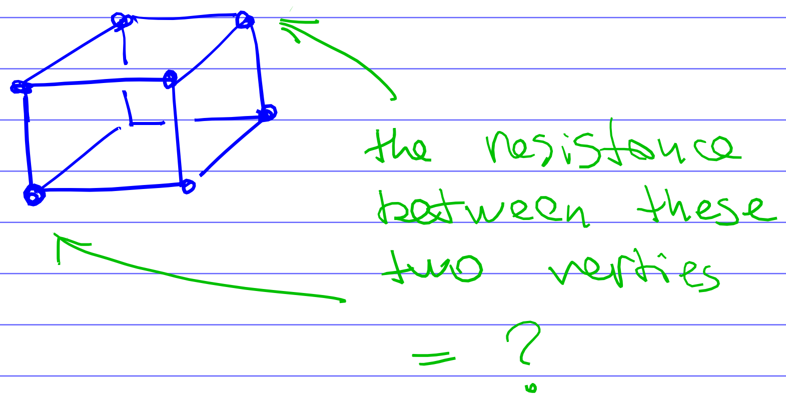
$$(x+y)(x+y+n)^{n-1} =$$

$$= \sum_{k=0}^{n} \binom{n}{k} x (x+k)^{k-1} y (y+n-k)^{n-k-1}.$$

(5) Prove the reciprocity formula for the number of spanning trees given in Lecture 21.

(6) Consider the electrical network given by the 1-skeleton of the n-dimensional hypercube, where each edge is a resistor of resistance 1 Ohm.

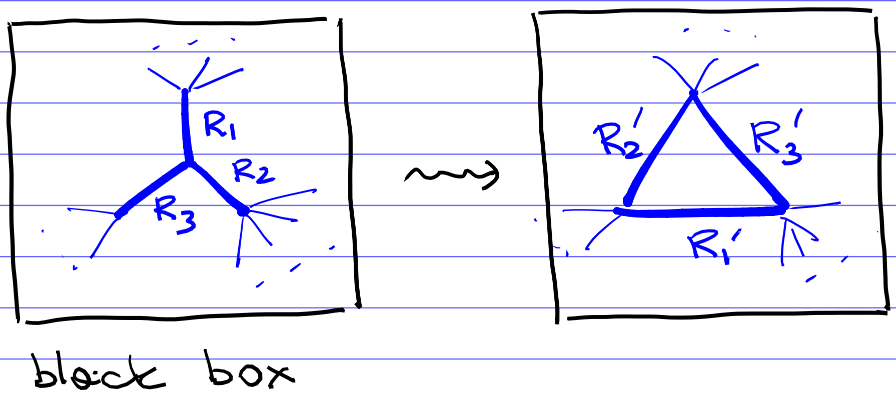
Calculate the resistance between two opposite vertices in this network.



(7) Find explicit expression for the $Y-\Delta$ transform

$$(R_1, R_2, R_3) \rightsquigarrow (R_1', R_2', R_3'),$$

see Lecture 25.



(8) Alice and Bob play the game, as described in the notes for Lecture 25.

Find the probability that Alice wins.

(9) Prove the lemma given on page 3 of Lecture 26 about the equivalence of the 3 descriptions (A), (B), (C) of parking functions.

(10) Find a bijection between parking functions $(f_1, \dots, f_n)$ and spanning trees of the complete graph K_{n+1}.

(11) Fix integers $n, k, \ell \geq 1$. $(f_1, \dots, f_n)$ is a <u>generalized parking function</u> if

- $f_1, \dots, f_n$ are positive integers
- the weakly increasing rearrangement $f_1' \leq f_2' \leq \dots \leq f_n'$ of the numbers $f_1, \dots, f_n$ satisfies:

$$
\begin{aligned}
f_1' &\leq \ell \\
f_2' &\leq \ell + k \\
f_3' &\leq \ell + 2k \\
f_4' &\leq \ell + 3k \\
&\dots\dots \\
f_n' &\leq \ell + (n-1)k
\end{aligned}
$$

Prove the formula for the number of such generalized parking functions given in Lecture 26.

(12) Fix a positive integer n.

Find the number of sequences $(i_1, i_2, \ldots, i_N)$, where $N = n \cdot (n-1)$, such that

- $i_1, i_2, \ldots, i_N \in \{1, \ldots, n\}$
- $i_{k+1} \neq i_k$, for any k, and $i_1 \neq i_N$.
- Any pair (i, j) $i, j \in [n]$, $i \neq j$ occurs exactly once among the pairs $(i_1, i_2), (i_2\ i_3), \ldots, (i_{N-1}, i_N), (i_N, i_1)$.

(13) Show that the polynomials

$$P_n(x) := \sum_{\substack{(f_1, \ldots, f_n) \\ \text{parking function}}} x^{\binom{n+1}{2} - (f_1 + \ldots + f_n)}$$

satisfy the recurrence relation

$$P_n(x) =$$

$$= \sum_{k=1}^{n} \binom{n-1}{k-1} (1 + x + x^2 + \ldots + x^{k-1}) P_{k-1}(x) P_{n-k}(x)$$

for $n \geq 1$. $\quad P_0(x) = 1$.

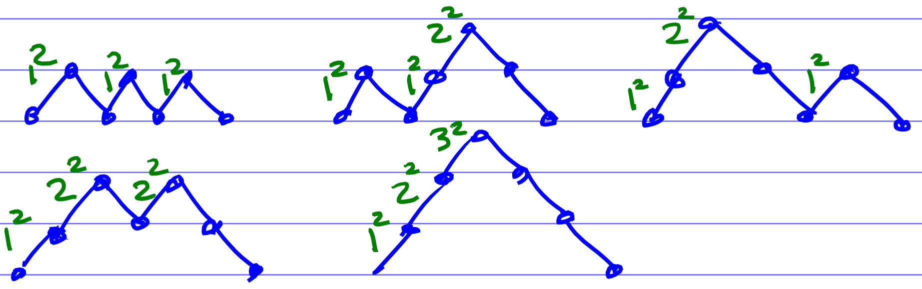
(14) Show that the numbers A_n of alternating permutations in S_n appear on the sides of the Euler - Bernoulli triangle, see Lecture 31.

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 0 & & 1 & & & & \\
 & & & 1 & & 1 & & 0 & & & \\
 & & 0 & & 1 & & 2 & & 2 & & \\
 & 5 & & 5 & & 4 & & 2 & & 0 & \\
0 & & 5 & & 10 & & 14 & & 16 & & 16 \\
 & & & & & \cdots & & & & &
\end{array}
$$

(15) Prove that the number A_{2n} of alternating permutations equals the weighted sum over Dyck paths:

$$A_{2n} = \sum_{\substack{P \text{ Dyck} \\ \text{path with} \\ 2n \text{ steps}}} \prod_{\substack{s \text{ is an} \\ \text{up step} \\ \text{in } P}} ht(s)^2 .$$

For example, for $n=3$, we have



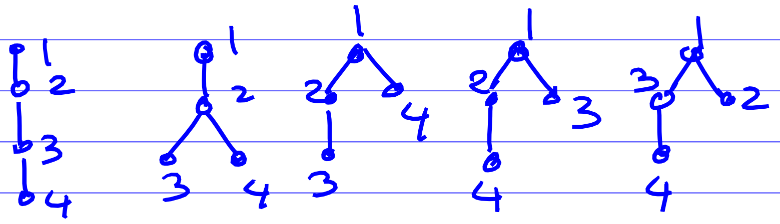
$$A_6 = 1^2 \cdot 1^2 \cdot 1^2 + 1^2 \cdot 1^2 \cdot 2^2 + 1^2 \cdot 2^2 \cdot 1^2 + 1^2 \cdot 2^2 \cdot 2^2 + 1^2 \cdot 2^2 \cdot 3^2$$

$$= 61.$$

(16) An <u>increasing 012-tree</u> is a labelled tree on vertices $1, 2, \ldots, n$ such that

- T is an <u>increasing</u> tree, i.e. the labels increase as we go away from vertex 1
- If we direct all edges of T away from vertex 1, then the outdegree of any vertex in T is at most 2.

For example, for $n=4$, there are 5 increasing 012-trees:



Prove that the number of increasing 012-trees on n vertices equals the number A_n of alternating permutations.

(17) Prove the identity for formal power series:

$$\sum_{n\geq 0} n!\, x^n =$$

$$= \cfrac{1}{1 - x - \cfrac{x^2}{1 - 3x - \cfrac{(2x)^2}{1 - 5x - \cfrac{(3x)^2}{1 - 7x - \cfrac{(4x)^2}{1 - \dots}}}}}$$

(18) (A) Show that the numbers A_n of alternating permutations satisfy the recurrence relation:

$$2 A_{n+1} = \sum_{k=0}^{n} \binom{n}{k} A_k A_{n-k}$$

for $n \geq 1$

$$A_0 = A_1 = 1$$

(B) Prove the identity

$$(x+y)^n =$$

$$= \sum_{k=0}^{n} \binom{n}{k} y (y+kz)^{k-1} (x-kz)^{n-k}$$

<u>Note</u>: parts (A) & (B) are not really related to each other. I grouped them together because each of them worths half a problem.

Compare part (B) with problem (4).