18.212 **Problem Set 2**

due Friday, April 9, 2021

Solve 5 (or more) problems.

① Prove Erdős - Szekeres Theorem:

Fix $m, n \geq 1$. Any permutation of size $\geq m \cdot n + 1$ either has an increasing subsequence with $m+1$ elements or a decreasing subsequence with $n+1$ elements.

(In class, we explained that Greene's Thm easily implies Erdős - Szekeres Thm.

Here you need to find a direct proof of Erdős - Szekeres Thm without using Greene's Thm or Schensted correspondence.)

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②. For $n \geq 1$ and $0 \leq k \leq n/2$, construct a bijection ρ between k-element subsets in $[n] := \{1, 2, .., n\}$ and $(n-k)$-element subsets in $[n]$ such that $\rho(I) \supseteq I$, for any k-element subset $I \subset [n]$.

(3) Let Π_n be the poset of all set partitions of $[n]$ ordered by refinement. Prove that Π_n is a lattice.

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(4) Give a complete proof of the Fundamental Theorem on Finite Distributive Lattices.

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(5) Let $n_1, \ldots, n_k \geq 1$ and $n = n_1 + \ldots + n_k$. For a permutation $w = w_1 \ldots w_n$ of the multiset

$$\{1^{n_1}, 2^{n_2}, \ldots, k^{n_k}\} = \{\underbrace{1, \ldots, 1}_{n_1}, \underbrace{2, \ldots, 2}_{n_2}, \ldots, \underbrace{k, \ldots, k}_{n_k}\},$$

define the inversion number

$$\mathrm{inv}(w) = \#\{1 \leq i < j \leq n \mid w_i > w_j\}.$$

Prove that the q-multinomial coefficient

$$\begin{bmatrix} n \\ n_1 \ldots n_k \end{bmatrix}_q := \frac{[n]_q!}{[n_1]_q! \ldots [n_k]_q!}$$

equals

$$\sum_{\substack{w \text{ perm.} \\ \text{of the} \\ \text{multiset } \{1^{n_1}, 2^{n_2}, \ldots, k^{n_k}\}}} q^{\mathrm{inv}(w)}$$

(6) Let $w \in S_n$ be a permutation. Let λ be the Schensted shape of λ (i.e. λ is the shape of the P-tableau and Q-tableau corresponding to λ).

(A) Prove that λ_1 is the size of a longest increasing subsequence in w.

(B) Prove that λ'_1 is the size of a longest decreasing subsequence in w.

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(7) (A) Give a non-recursive construction of the Fibonacci lattice $\mathbb{F}$.

(B) Prove that the Fibonacci lattice $\mathbb{F}$ is a lattice.

(8) Let $\lambda = (\lambda_1, \ldots, \lambda_n) \subseteq n \times n$ be a Young diagram that fits inside the $n \times n$ square. Let $\lambda' = (\lambda'_1, \ldots, \lambda'_n)$ be its conjugate.

Prove that the multiset of numbers $\lambda_n, \lambda_{n-1} - 1, \lambda_{n-2} - 2, \ldots, \lambda_1 - n + 1$ coincides with the multiset $\lambda'_n, \lambda'_{n-1} - 1, \lambda'_{n-2} - 2, \ldots, \lambda'_1 - n + 1$.

(In class, we showed that $\lambda_n \cdot (\lambda_{n-1} - 1) \cdot \ldots \cdot (\lambda_1 - n + 1)$ equals the number of placements of n non-attacking rooks inside the Young diagram λ.)

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(9) Let f_λ be the number of SYT's of shape λ.

Find an explicit formula for

$$\sum_{\lambda \vdash n} f_\lambda .$$

(The formula will involve a sum of a certain closed expression.)

(10) Construct a bijection between perfect matchings in the complete graph K_{2n} and the set of oscillating tableaux $(\lambda^{(0)}, \dots, \lambda^{(2n)})$ such that

- $\lambda^{(0)}, \dots, \lambda^{(2n)}$ are Young diagrams.
- $\lambda^{(i)}$ and $\lambda^{(i+1)}$ are obtained from each other by adding or removing a single box, for any i.
- $\lambda^{(0)} = \lambda^{(2n)} = \emptyset$.

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(11) (A) Prove that the number of oscillating tableaux $(\lambda^{(0)}, \dots, \lambda^{(n)})$ such that

- $\lambda^{(0)}, \dots, \lambda^{(n)}$ are Young diagr.
- $\lambda^{(i)}$ and $\lambda^{(i+1)}$ differ by a single box, $\forall i$
- $\lambda^{(0)} = \emptyset$

equals the number of involutions in S_n with all 2-cycles colored in 2 colors.

(B) Give an explicit formula (involving a summation) for this number.

(12) Construct a bijection between partitions of n with odd parts and partitions of n with distinct parts.

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(13) Prove the following q-binomial formula:

$$(1+x)(1+xq)(1+xq^2)\dots(1+xq^{n-1}) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2} x^k .$$

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(14) Prove the following identity:

$$\begin{bmatrix} 2n \\ n \end{bmatrix}_q = \sum_{k=0}^{n} q^{k^2} \left(\begin{bmatrix} n \\ k \end{bmatrix}_q \right)^2 .$$

(15) Prove the identity

$$\cfrac{1}{1-\cfrac{qx}{1-\cfrac{q^2x}{1-\cfrac{q^3x}{1-\cdots}}}} = \frac{\sum_{n\geq 0} q^{-\frac{n\cdot(n+1)}{2}} \begin{bmatrix} 2n+1 \\ n \end{bmatrix}_q x^n}{\sum_{n\geq 0} q^{-\frac{n\cdot(n+1)}{2}} \begin{bmatrix} 2n \\ n \end{bmatrix}_q x^n}.$$

(The L.H.S. is called the Ramanujan continued fraction.)

(16) Prove Lemma on page 12 in Lecture 17.

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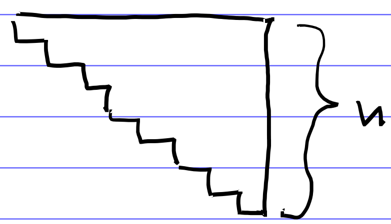
(17) Let $b_r = b_{r,n}$ be the number of shifted Young diagrams with r boxes that fit inside the "shifted staircase" shape:



Prove the unimodality

$$b_0 \le b_1 \le b_2 \le \ldots \le b_\ell \ge b_{\ell+1} \ge \ldots \ge b_M,$$

where $M = n \cdot (n+1)/2$ and $\ell = \lfloor \frac{M}{2} \rfloor$.