18.212 **Problem Set 1**

due Friday, March 12, 2021

It is enough to solve 5 problems.

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① Consider the random walk on the segment $[0, N] := \{0, 1, \ldots, N\}$ of the integer line such that

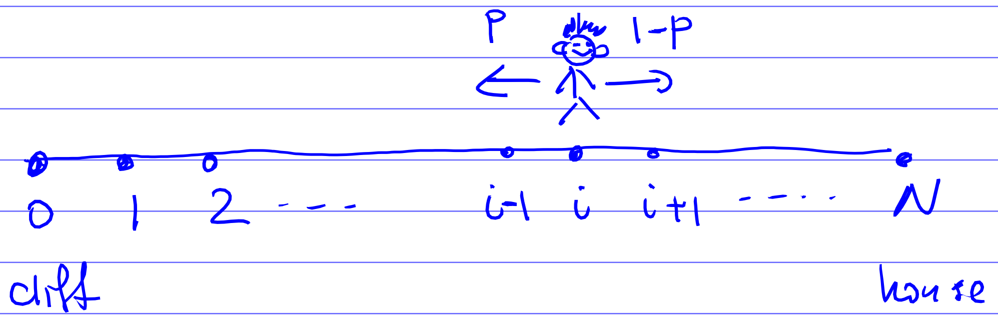
- we can go from the position i either to $i-1$ or to $i+1$ with the probabilities

$$\mathrm{Prob}(i, i-1) = p \quad \text{and}$$

$$\mathrm{Prob}(i, i+1) = 1-p,$$

where $i \in \{1, 2, \ldots, N-1\}$ and p is some fixed constant $0 \le p \le 1$.

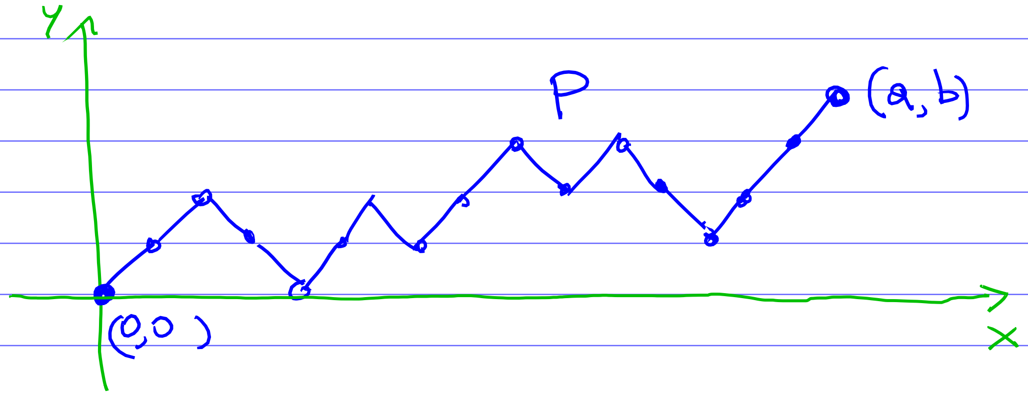
- a random walk stops when it reaches either 0 (cliff) or N (house).



Find the probability p_i that a random walk starting in the position i reach the position N (before reaching 0).

(2) Find an explicit formula for the number of lattice paths P (similar to Dyck paths) such that

- P starts at $(0,0)$ and ends at (a,b)
- P has steps given by the vectors $(1,1)$ ("up" steps) and $(1,-1)$ ("down" steps)
- P always stays in the upper half plane $\{(x,y) \in \mathbb{R}^2 \mid y \geq 0$



Here a, b are any nonnegative integers such that $a \geq b$ and $a+b$ is even.

For example, if $(a,b) = (2n, 0)$ then such paths P are exactly Dyck paths with $2n$ steps.

③ Prove the lemma from Lecture 2 about cyclic shifts.

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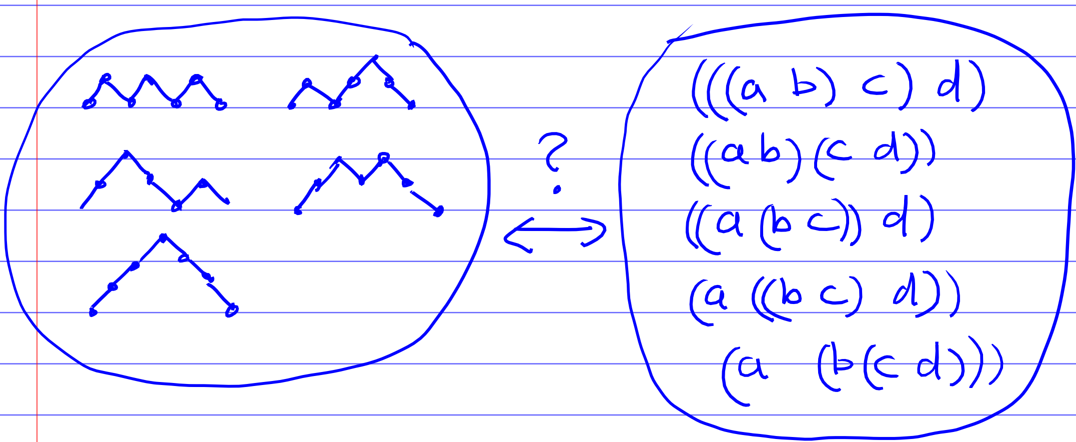
④ Construct a bijection between Dyck paths with $2n$ steps and valid parenthesizations of $n+1$ letters.



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⑤ Prove that the stack-sortable permutations are exactly the 231-avoiding permutations.

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⑥ Prove that the queue-sortable permutations are exactly the 321-avoiding permutations

(7) Prove that the Catalan number C_n equals:

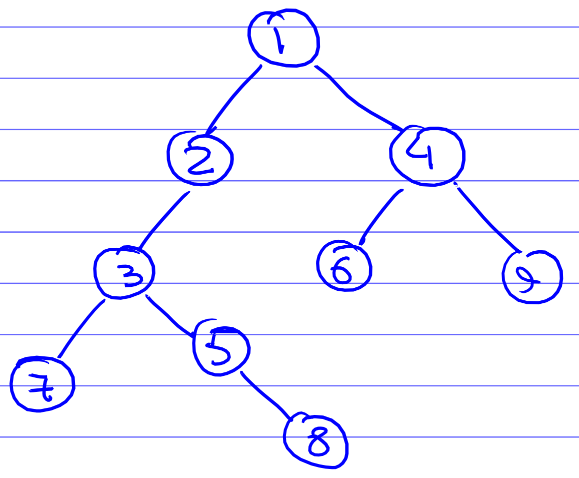
(A) the number of 231-avoiding permutations in S_n

(B) the number of 321-avoiding permutation in S_n

(8) Find the number of permutations in S_n which are both 231-avoiding and 321-avoiding.

(9) Let T be a binary tree with n vertices. Define a "standard tableau" of shape T as labelling of the vertices of T by numbers $1, 2, \ldots, n$ such that the label of any vertex v is greater than the label of any child of T.

Let f_T be the number of such "standard tableaux" of shape T.



a "standard tableau" of shape T

For a vertex v of T, define the "hook length" $h(v)$ as the number of all descendants of v, including the vertex v itself.

Prove the following version of hook length formula:

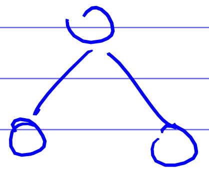
$$\boxed{f_T = \frac{n!}{\prod\limits_{v \text{ vertex of } T} h(v)}}$$

(10) Let f_T be the number defined in problem 9.

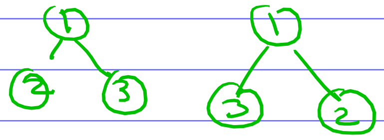
Find a closed formula for the sum

$\sum_T f_T$ over all binary trees with n vertices.

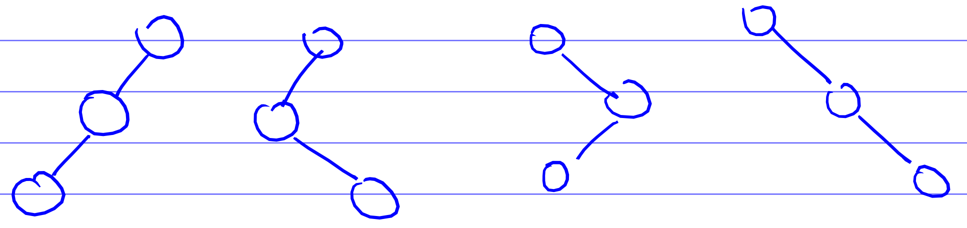
For example, for $n=3$, we have



has 2 standard tableaux:



and



each of these binary trees has 1 standard tableau.

In total we get, $2+1+1+1+1=6$.

(11) Recall that the Bell number $B(n)$ is the total number of set partitions of $[n]$. Show that $B(n)$ equals the number of set partitions π of $[n+1]$ satisfying the condition: there is no i such that both i and $i+1$ belong to the same block of π.

$n = 3$

	$\overset{?}{\longleftrightarrow}$	
$(1\,2\,3)$		$(1\,3 \mid 2\,4)$
$(1\,2 \mid 3)$		$(1\,3 \mid 2 \mid 4)$
$(1\,3 \mid 2)$		$(1\,4 \mid 2 \mid 3)$
$(2\,3 \mid 1)$		$(1 \mid 2\,4 \mid 3)$
$(1 \mid 2 \mid 3)$		$(1 \mid 2 \mid 3 \mid 4)$
all set partitions of $[3]$		set partitions of $[4]$ whose blocks are not allowed to contain adjacent entries i & $i+1$.

(12) Prove that the number of non-crossing set partitions of $[n]$ equals the Catalan number C_n

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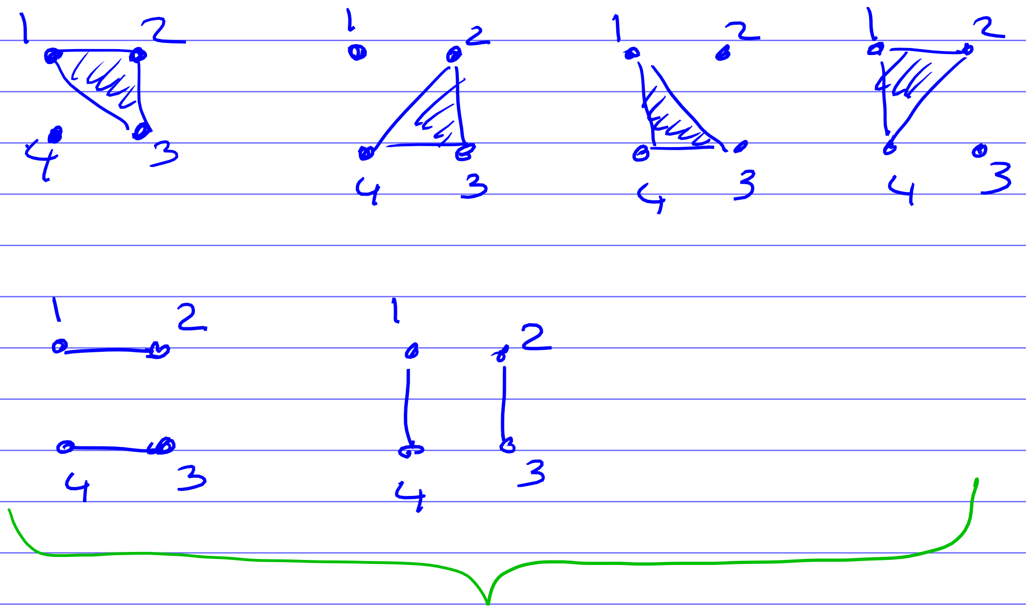
(13) Prove that the number of non-crossing set partitions of $[n]$ with exactly k blocks equals the Narayana number $N(n,k)$.

(In class, we proved a similar claim about non-nesting set partitions).

Example, for $n=4$, $k=2$



$N(4,3) = 6$ non-crossing set partitions of $[4]$ with 2 blocks.

Recall that we defined the Narayana number $N(n,k)$ as the number of Dyck paths with $2n$ steps and k peaks.

(14) Prove the following symmetry of the Narayana numbers:

$$N(n,k) = N(n, n-k+1)$$

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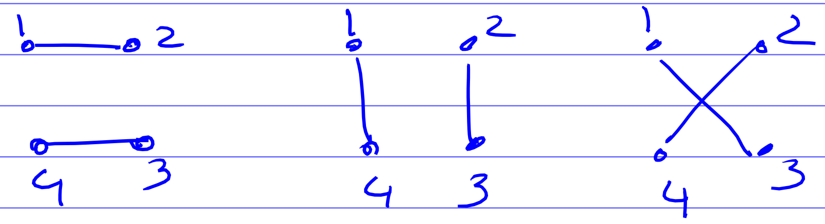
(15) Let M_n be the number of perfect matchings on n labelled vertices.

(A <u>perfect matching</u> is a simple graph such that degrees of all vertices equal to 1.)

For example, $M_4 = 3$



3 perfect matchings on 4 labelled vertices.

(A) Find the exponential generating function

$$\sum_{n \geq 0} M_n \frac{x^n}{n!}$$

(B) Find a closed expression for M_n.

(16) Show maj (the major index) and inv (the number of inversions) are equidistributed statistics on S_n

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(17) Show that exc(w) (the number of exceedances) and wexc(w) − 1 (the number of weak exceedances minus 1) are equidistributed statistics on S_n

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(18) Prove the following recurrence relations for the signless Stirling numbers of the first kind $c(n,k)$, the Stirling numbers of the second kind $S(n,k)$, and the Eulerian numbers $A(n,k)$:

(A) $c(n,k) = c(n-1,k-1) + (n-1)\,c(n-1,k)$

(B) $S(n,k) = S(n-1,k-1) + k\,S(n-1,k)$

(C) $A(n,k) = (n-k)\,A(n-1,k-1) + (k+1)\,A(n-1,k)$.