18.212 Lecture 37 05/19/2021

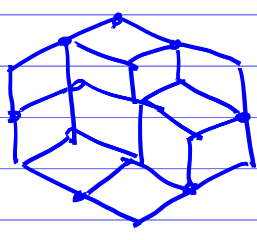
The last lecture in this course

This semester we discussed many topics. But all this is a tiny fraction of modern combinatorics. I tried to show you glimpses of popular themes. If you found this material interesting, you shoud definitely continue studying it on a deeper level...

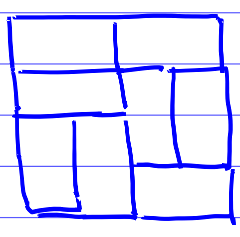
Let me mention a few more things...

Domino Tilings

(Last week we discussed rhombus tilings.)



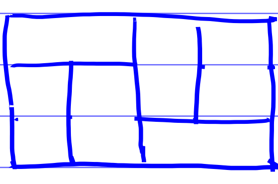
a rhombus tiling



a domino tiling

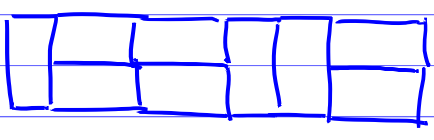
Def. A domino tiling is a way to subdive some region on the plane (typically, an $m \times n$ rectangle) into dominos (1×2 or 2×1 rectangles).

Example $m=3$, $n=4$



a domino tiling of 3×4 rectangle

$m=2$, $n=9$

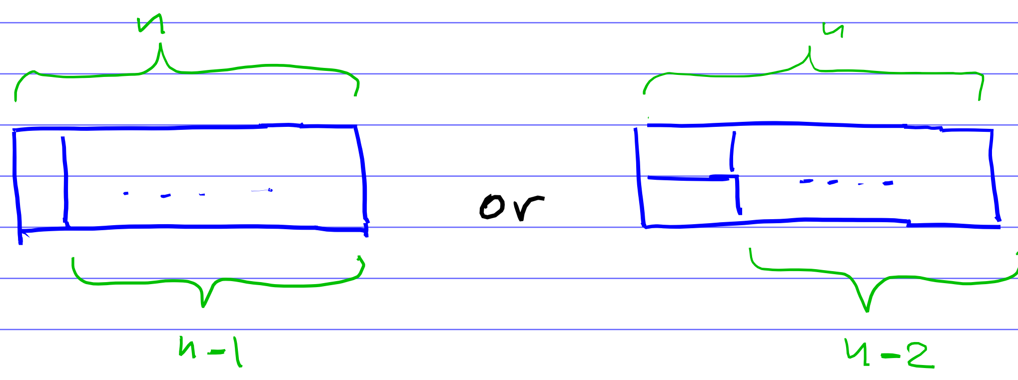


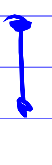
a domino tiling of 2×9 rectangle

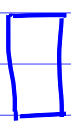
Theorem. The number of domino tilings of an $2 \times n$ rectangle equals the Fibonacci number F_{n+1}.

Proof $F_{n+1} = F_n + F_{n-1}$

It is easy to see this recurrence for # domino tilings of a $2 \times n$ rectangle:



Also the initial conditions are correct: $n=0$  $1 = F_1$ tiling

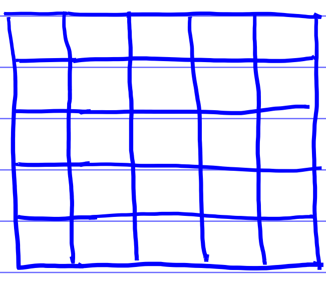
$n=1$  $1 = F_2$ tiling

□

Clearly, we can tile an $m \times n$ rectangle by dominos iff $m \cdot n$ is even.

What about an $m \times n$ rectangle where both m & n are odd?

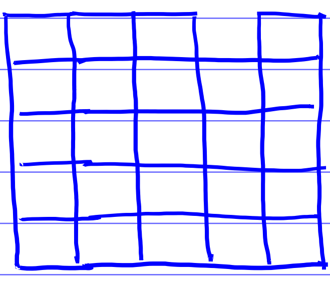
Example $m = n = 5$



5×5 square

We cannot subdivide the 5×5 square into dominos because it has the odd number 5^2 of boxes.

How about the region obtained by removing a sigle box, e.g.

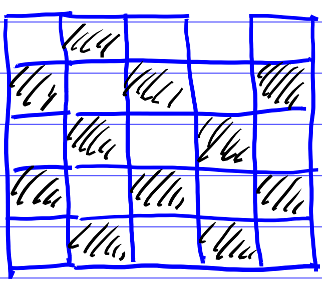


Can we tile this region by dominos?

$5^2 - 1 = 24$ boxes

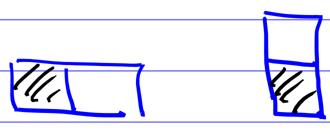
Answer: No

Lets color all boxes in black & white like a chessboard:



11 black boxes

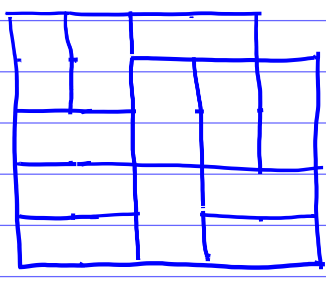
13 white boxes

But any domino () should contain exactly one black box and exactly one white box.

So we cannot tile this region by dominos?

In order to have the same numbers of black & white boxes, the color of the removed box should be the same as the color of a corner box.

Example



We can tile the 5×5 square without a corner box by dominos

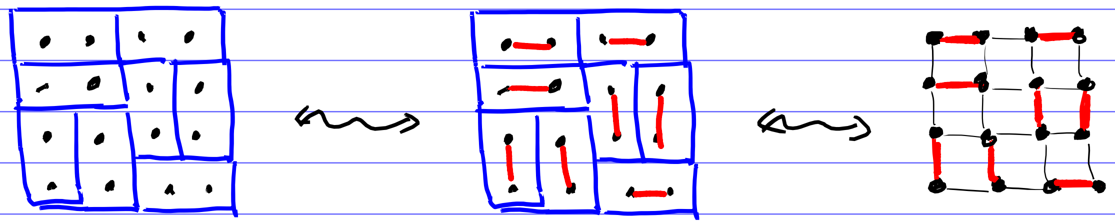
Can we find the number domino tilings?

Perfect matchings

Domino tilings & rhombus tilings are special cases of <u>perfect matchings</u>.

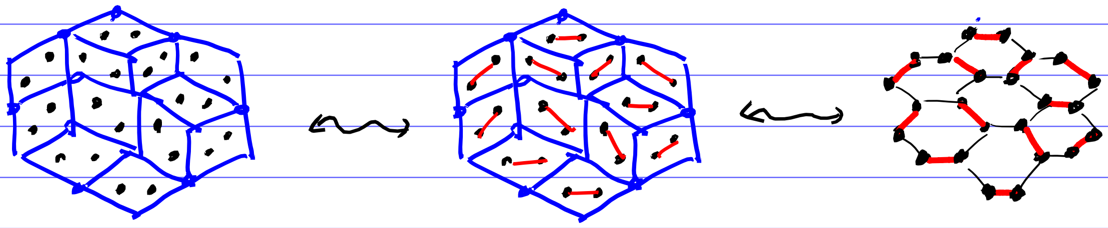
- domino tilings $\xleftrightarrow{\text{bij.}}$ perfect matchings in grid graphs

Example



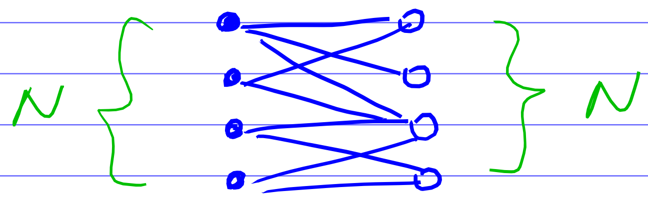
- rhombus tilings $\xleftrightarrow{\text{bij.}}$ perfect matching in honeycomb graph

Example

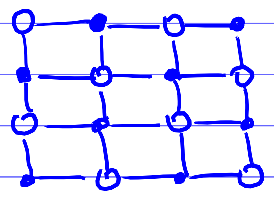


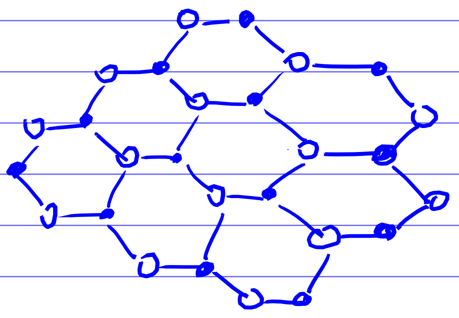
How to count # perfect matchings in a graph G ?

Assume that G is a <u>bipartite</u> graph with the some numbers N of vertices of both colors, i.e. $G \subset K_{N,N}$



Notice that square grids and honeycombs graphs are bipartite:



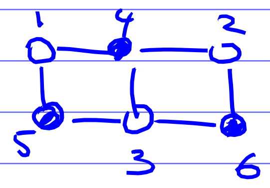


Let A be the <u>adjacency matrix</u> of G.

It is an $2N \times 2N$ matrix of the form:

$$A = \begin{array}{cc} \begin{array}{c} N \\ N \end{array} & \left[\begin{array}{c|c} 0 & B \\ \hline B^T & 0 \end{array}\right] \\ & \begin{array}{cc} N & N \end{array} \end{array}$$

<u>Example</u> G =



(First, label vertices of one color by 1, 2, .., N then label vertices of the other color by N+1, N+2, ..., 2N)

$$A = \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right]$$

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Determinant & Permanent

$C = (c_{ij})$ an $N \times N$ matrix

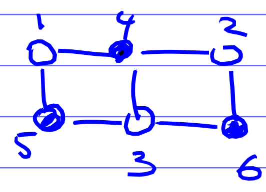
$$\det(C) := \sum_{w \in S_n} (-1)^{\ell(w)} \prod_{i=1}^{N} c_{i w_i}$$

$$\operatorname{per}(C) := \sum_{w \in S_n} \prod_{i=1}^{N} c_{i w_i}$$

The permanent per(C) is given by the same formula as det(C) but without signs.

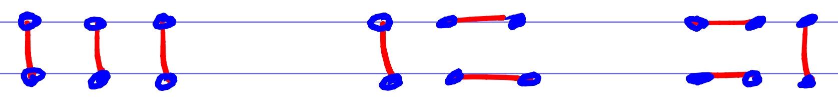
Clearly, # perfect matchings of G = per(B).

Example G =



$$\operatorname{per}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} = 3$$

3 perfect matchings



Q: Does this help us?

A: Not much.

It is (relatively) easy to calculate det(C), because we can use many tools from linear algebra:

- eigenvalues & eigenvectors
- row & column operations
- elimination

etc.

But it is hard to calculate the permanent. Basically, there are no any general exact formulas for per(C).

However, in one case we can calculate the permanent.

If G is a planar graph, then

$$\operatorname{per}(B) = \det(\tilde{B})$$

where $\tilde{B}$ is a certain matrix obtained from B but putting signs in front of some entries.

Theorem (Kastelegn)

Let G be a planar bipartite graph (with a particular choice of drawing on the plane).

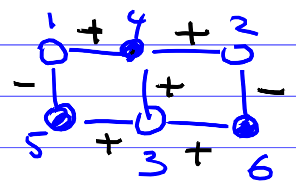
- It is possible to assign signs to edges of G (and obtain a signed matrix $\tilde{B}$) such that:

  For any face of G, an odd number of edges has the "$-$" sign.

- Then the number of perfect matchings of G equals

$$\operatorname{per}(B) = |\det(\tilde{B})|$$

Example



$$B = \begin{array}{c|ccc} & 4 & 5 & 6 \\ \hline 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \\ 3 & 1 & 1 & 1 \end{array}, \quad \tilde{B} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\det(\tilde{B}) =$$

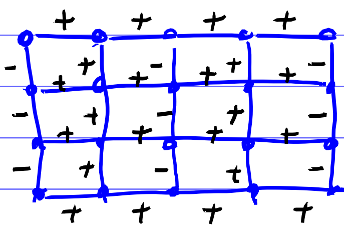
$$-1\cdot(-1)\cdot 1 - (-1)\cdot 1\cdot 1 + (-1)\cdot(-1)\cdot 1$$

$$= 1 + 1 + 1$$

← all signs disappear

It is not hard to prove this theorem by induction on # faces in G, by removing a boundary face in G.

For an $m \times n$ grid (such that $m \cdot n$ is even) we can assign the signs like this:



- all horizontal edges have "+" signs
- the signs of vertical edges alternate in rows.

Actually, Kasteleyn proved a more general result for *any* planar graph G (not necessarily bipartite).

Kasteleyn's Theorem (1967)

G a planar graph drawn on the plane.

- It is possible to orient the edges of G so that every face of G has an *odd* number of edges oriented clockwise.

- Let $\tilde{A}$ be the *signed* adjacency matrix of this oriented graph:

$$\tilde{A}_{ij} = \begin{cases} 1 & \text{if } i \to j \\ -1 & \text{if } j \to i \\ 0 & \text{otherwise} \end{cases}$$

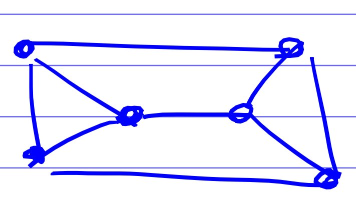
Then

$$\#\{\text{perfect matchings of } G\} = \sqrt{\det(\tilde{A})}$$

this is the Pfaffian of $\tilde{A}$. It is defined for any antisymmetric matrix

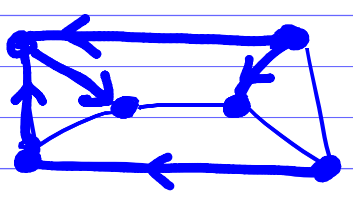
Example

G =

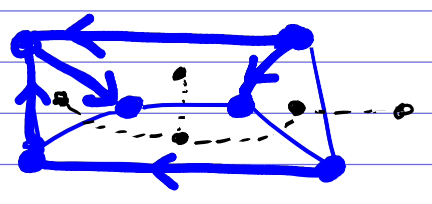


Let's construct its orientation that satisfies the condition.

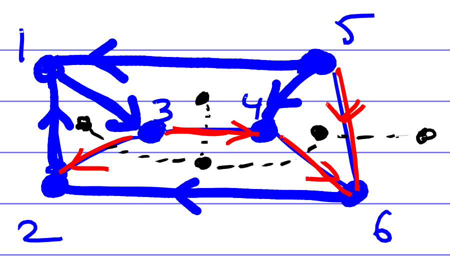
1. Pick a spanning tree of G and orient its edges arbitrarily:



2. The remaining edges correspond to a spanning tree of the dual graph G^*



There is a *unique* way to orient these edges one by one starting at the leaves of this tree (of G^*) so that the condition holds.

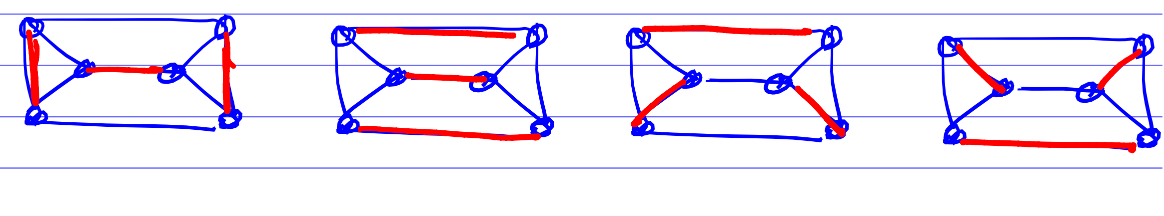


The corresponding oriented adjacency matrix is

$$\tilde{A} = \begin{bmatrix} 0 & -1 & 1 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 & -1 & 0 \end{bmatrix}$$

$\det \tilde{A} = 4^2$

There are 4 perfect matchings



For an $m \times n$ grid graph, it is possible to explicitly calculate the eigenvalues of matrix $\tilde{B}$ in Kasteley's Theorem, and get the following explicit formula.

__Theorem__ (Kastelegn 1961)

Assume that n is even.

domino tilings of an $m \times n$ rectangle equals

$$\prod_{k=1}^{n/2} \prod_{l=1}^{\lfloor m/2 \rfloor} \left(4\cos^2 \frac{\pi k}{n+1} + 4\cos^2 \frac{\pi l}{m+1}\right).$$

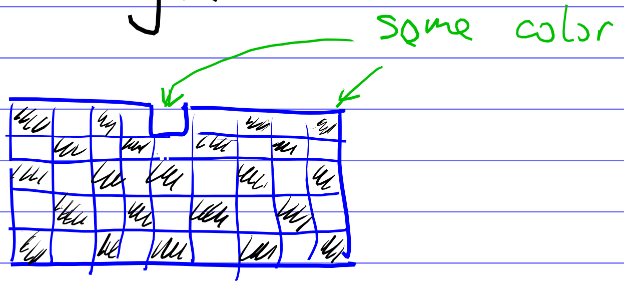
How about an $m \times n$ rectangle where both m & n are odd?

<u>Theorem</u> (Temperley, 1974)

Suppose that m & n are both odd $m = 2k+1$, $n = 2\ell+1$.

Consider a region R obtained from a $m \times n$ rectangle by removing a single box b s.t.

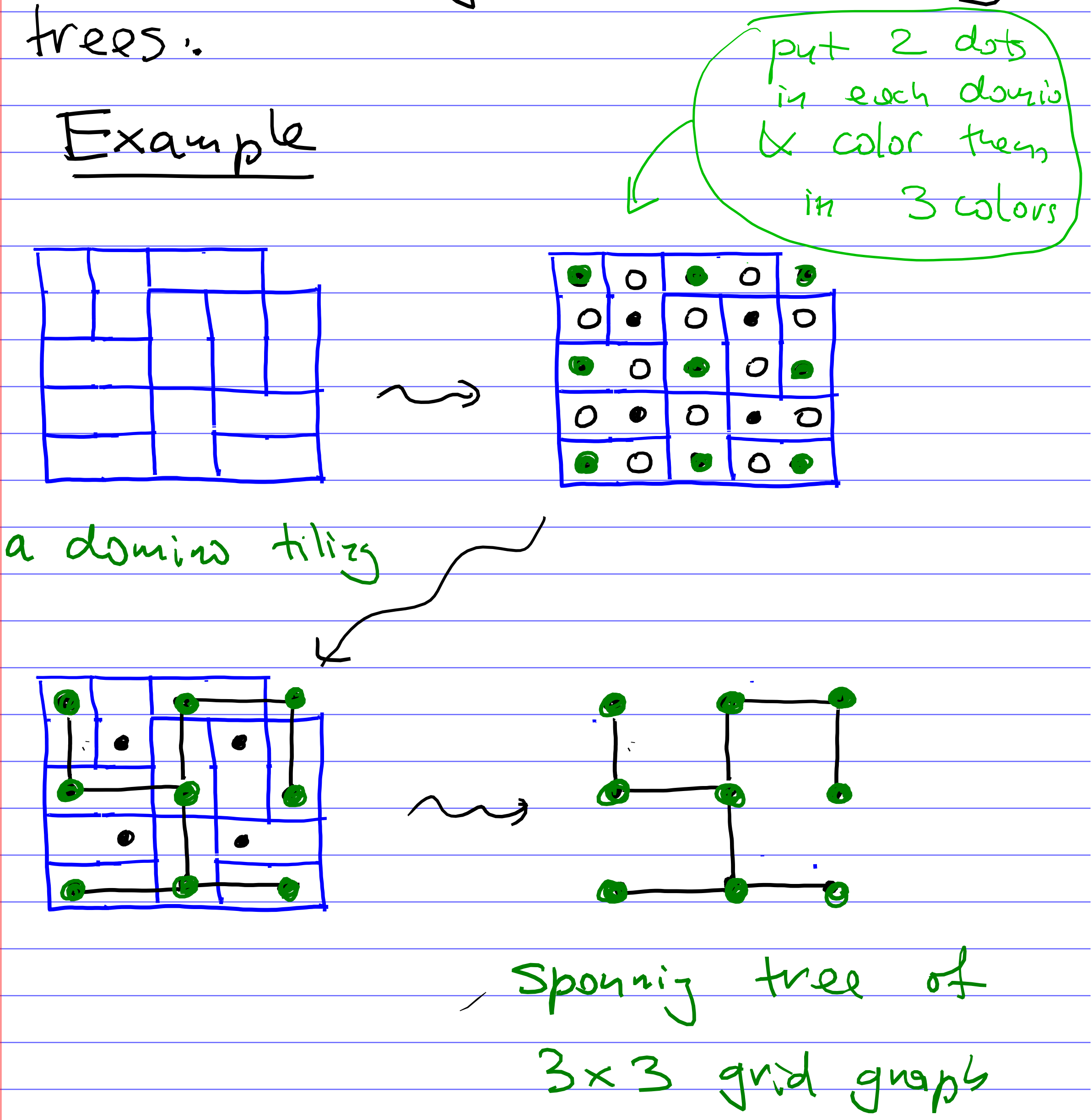
- b is on the <u>boundary</u> of the rectangle
- b has the <u>same color</u> as corners of the rectangle

$R =$ 

Then # domino tilings equals # spanning trees of the $k \times \ell$ grid graph.

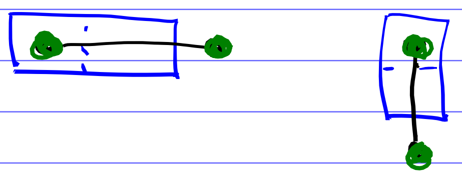
Proof Let's construct a bijection between domino tilings & spanning trees.

Example



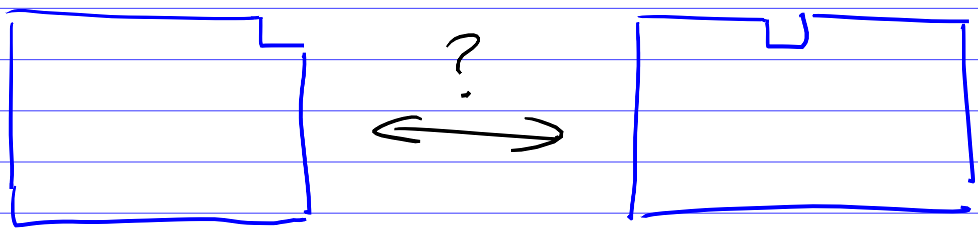
Rule for edges:

Connect the green dots like this:



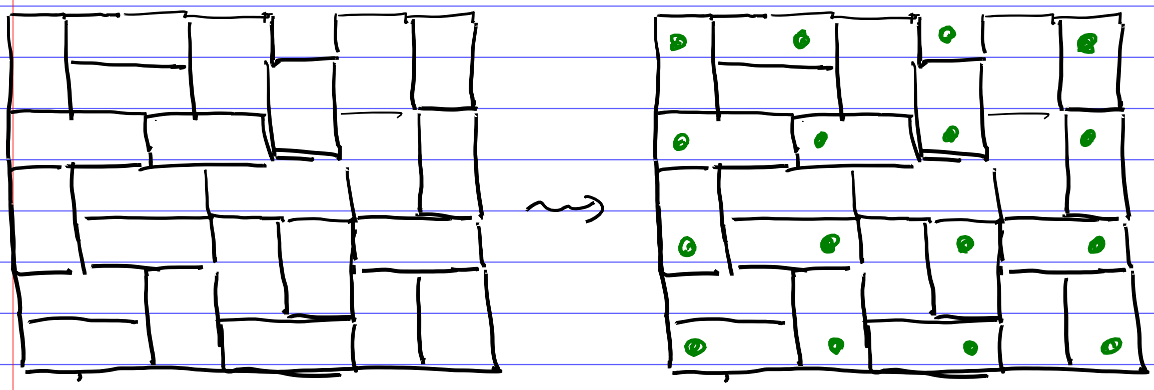
Claim: this construction gives a bijection between domino tilings & spanning trees.

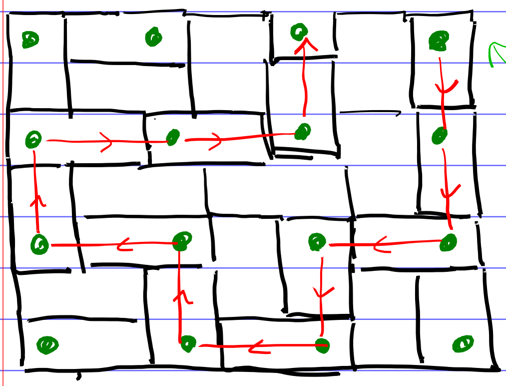
Observation If we remove any other box b of the same color on the boundary of the m×n rectangle, we get the same number of domino tilings.



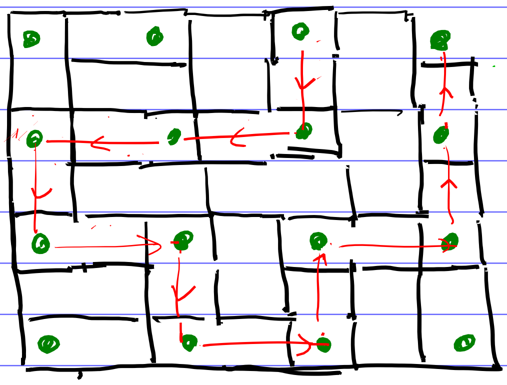
a bijection between domino tilings

Example



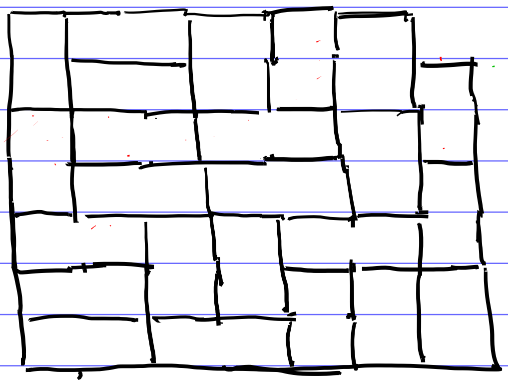


Construct the path starting at this corner using the previous rule.



The path should finish at the missing boundary box.

slide all dominos in this path by 1 box



The resulting domino tiling.

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THE END