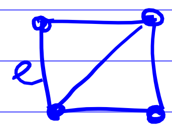
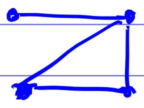
18.212 Lecture 36 05/17/2021

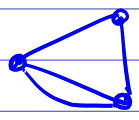
Tutte polynomial

Last time: Deletion - contraction

$G \rightsquigarrow G \setminus e$ (deletion of edge e)

$G \rightsquigarrow G / e$ (contraction of edge e)

$G =$  $\rightsquigarrow$ $G \setminus e =$ 

$\rightsquigarrow$ $G / e =$ 

There are several invariants* of graphs that satisfy (a version) of deletion – contraction recurrence:

$$f(G) = f(G \setminus e) + f(G/e)$$

or "–"

- # acyclic orientations of G

$$AO(G) = AO(G \setminus e) + AO(G/e)$$

- chromatic polynomial $\chi_G(t)$

$$\chi_G(t) = \chi_{G \setminus e}(t) - \chi_{G/e}(t)$$

- # spanning trees $ST(G)$:

$$ST(G) = ST(G \setminus e) + ST(G/e).$$

* Here the word "invariant" means that all these numbers and polynomials don't depend on a choice of ordering of the vertices of G.

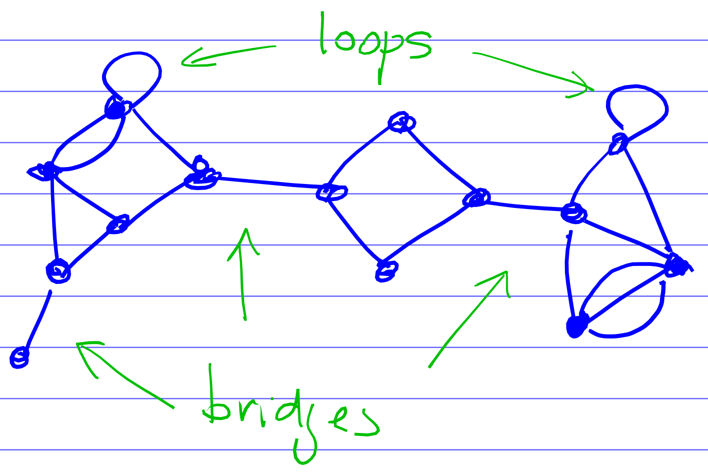
Is there the most general graphical invariant that satisfies the deletion-contraction recurrence?

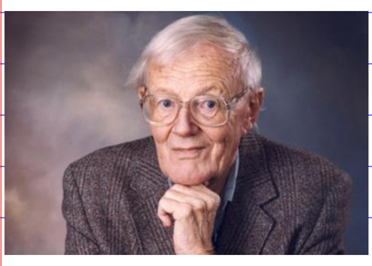
The Tutte polynomial $T_G(x,y)$

G - an undirected graph (we allow multiple edges and loops.)

An edge e in G is called a bridge (a.k.a. isthmus) if $G \setminus e$ has more connected component than G.

Example





William Thomas Tutte (1917-2002)

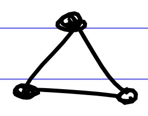
Theorem (Tutte) There exists a unique polynomial $T_G = T_G(x,y)$, with positive integer coefficients, such that

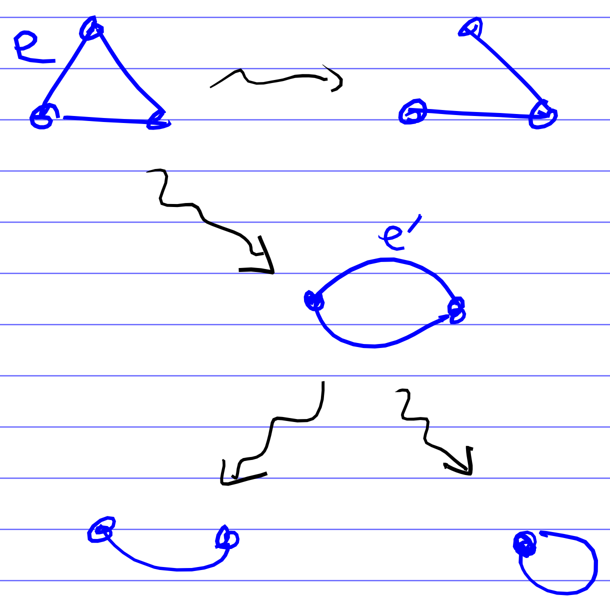
- For any edge e in G, which is not a loop nor a bridge, we have

$$T_G = T_{G \setminus e} + T_{G/e}$$

- If G has a bridges and b loops (and no other edges), then

$$T_G = x^a y^b.$$

Example. $G = K_3 =$ 



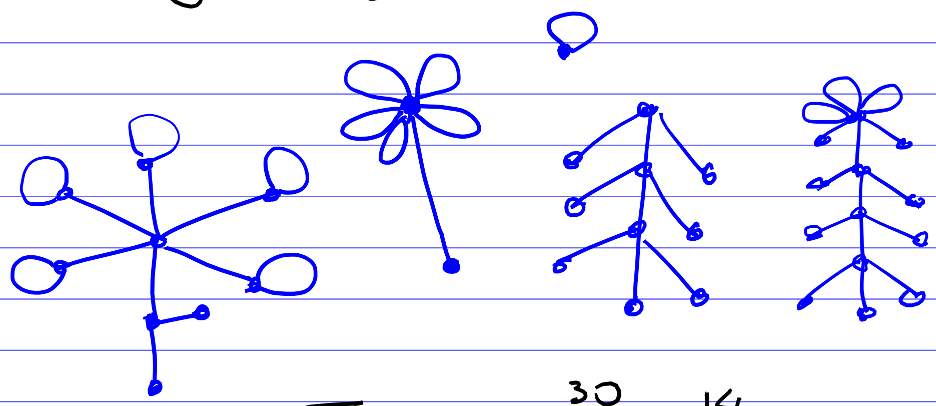
$$T_{K_3} = T + T$$

$$= x^2 + (T + T)$$

$$= x^2 + x + y.$$

All other graphical invariants we discussed last time ($\chi_G(t)$, $AO(G)$, $ST(G)$) are special values of the Tutte polynomial

Remark. The claim about uniqueness of $T_G(x,y)$ is clear. The deletion-contraction recurrence allows us to express $T_G(x,y)$ in terms of Tutte polynomials of graphs that consist only of bridges & loops, which are $x^{\#\text{bridges}} \, y^{\#\text{loops}}$.



$$T_G = x^{30} y^{14}$$

a graph that consists only of bridges & loops: a forest with some loops

But in order to prove existence of $T_G(x,y)$, we need to show that, if we do deletion-contraction in a different way, we obtain the same polynomial $T_G(x,y)$.

Possible approaches:

- induction on $|E|$
- give a non-recursive formula for $T_G(x,y)$, and show that it satisfies deletion-contraction

Whitney's corank - nullity formula

Theorem. For a graph $G=(V,E)$,

$$T_G(x,y) = \sum_{A \subseteq E} (x-1)^{r(E)-r(A)} (y-1)^{|A|-r(A)},$$

where the sum is over all subsets A of the edge set E (i.e. all subgraphs $H \subseteq G$).

$$r(E) := \text{the rank of } E = |V| - \#\left\{\begin{matrix}\text{connected}\\ \text{components}\\ \text{of } G\end{matrix}\right\}$$

$= \text{maximal number of edges in a forest } F \subseteq G.$

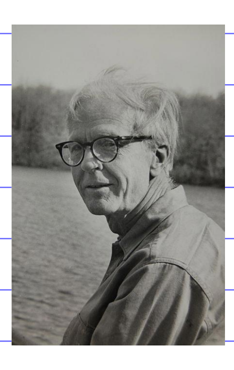
- $r(E)-r(A)$ is called the corank of A

  $= \#\{\text{connected components of } A\} - \#\{\text{connected components of } E\}$

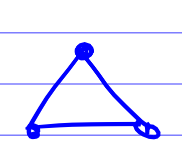
- $|A|-r(A)$ is called the nullity of A (a.k.a the cyclomatic number)

  = the minimum number of edges we need to remove from A to get a forest.

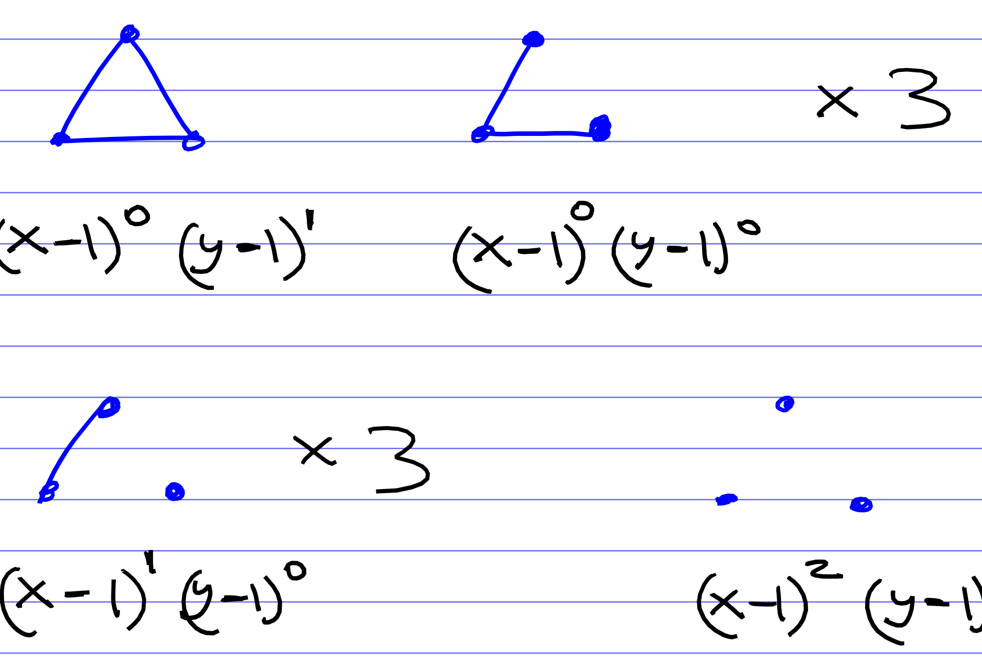
Remark The R.H.S. of this formula is Whitney's original definition from 1932.



Hassler Whitney
(1907 – 1989)

Example $G =$ 

It has 8 subgraphs:



$(x-1)^0 (y-1)^1$ $\quad$ $(x-1)^0 (y-1)^0$ $\times 3$

$(x-1)^1 (y-1)^0$ $\times 3$ $\quad$ $(x-1)^2 (y-1)^0$

$$T_{K_3}(x,y) = (x-1)^0(y-1)^1 + 3(x-1)^0(y-1)^0 + 3(x-1)^1(y-1)^0 + (x-1)^2(y-1)^0$$

$$= (y-1) + 3 + 3(x-1) + (x-1)^2$$

$$= x^2 + x + y.$$

In order to prove both theorems (the existance & uniqueness of $T_G(x,y)$ & Whitney's formula), it is enough to show that the R.H.S. of Whitney's formula satifies the deletion-contraction recurrence.

This can be done.

Whitney's formula gives a non-recursive expression for T_G. But this formula is not subtraction-free.

Ideally, we would like to find a non-recursive subtraction-free formula for $T_G(x,y)$.

We need:

- Find a combinatorial interpretation for $T_G(1,1)$.
- Define two statistics $a(T)$ & $b(T)$ on this set such that

$$T_G(x,y) = \sum x^{a(T)} y^{b(T)}.$$

Actually,

Lemma. If G is a connected graph, then

$$T_G(1,1) = \#\left\{\begin{matrix}\text{spanning trees}\\ \text{of } G\end{matrix}\right\}$$

In general, $T_G(1,1) = \#$ forests $F \subseteq G$ such that each connected component of F is a spanning tree of a connected component of G.

Proof (Assuming we already proved the existance & uniqueness thm). It is easy to see # spanning forests satisfies the deletion-contract. & the initial conditions for $T_G(1,1)$. $\square$

Lemma If G has connected components $G_1, \dots, G_k$, then

$$T_G(x,y) = \prod_{i=1}^{k} T_{G_i}(x,y).$$

Proof Also easy to prove by induction using deletion-contraction. $\square$

So we need to find two statistics on spanning trees of a connected graph G that produce the Tutte polynomial $T_G(x,y)$.

Internal & external activities

WLOG, assume that $G=(V,E)$ is a connected graph.

Fix a total ordering of the set E of edges of G.

Let $T \subset G$ be a spanning tree of G.

Definition (1) An edge $e \in T$ is called internally active if $\nexists$ smaller edge $e' < e$ s.t. $(T \setminus \{e\}) \cup \{e'\}$ is a spanning tree.

(2) An edge $f \in G \setminus T$ is called externally active if $\nexists$ smaller edge f' s.t. $(T \cup \{f\}) \setminus \{f'\}$ is a spanning tree.

Let

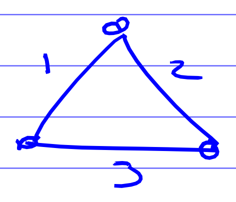
$$\mathrm{int}(T) := \#\left\{\begin{matrix}\text{internally active}\\ \text{edges w.r.t. tree } T\end{matrix}\right\}$$

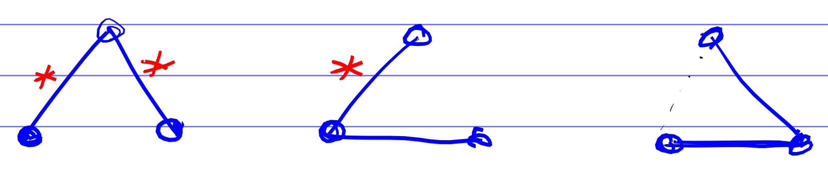
$$\mathrm{ext}(T) := \#\left\{\begin{matrix}\text{externally active}\\ \text{edges w.r.t. } T\end{matrix}\right\}$$

Theorem. (Tutte)

$$T_G(x,y) = \sum_{\substack{T \text{ is a}\\ \text{spanning tree}\\ \text{of } G}} x^{\mathrm{int}(T)}\, y^{\mathrm{ext}(T)}$$

Remark. Clearly $\mathrm{int}(T)$ and $\mathrm{ext}(T)$ depend on a choice of total order of edges in G. But the polynomial $T_G(x,y)$ does not depend on this choice.

Example $G =$ 



int = 2   int = 1   int = 0

ext = 0   ext = 0   ext = 1

$$T_{K_3} = x^2 + x + y.$$

Sketch of proof. We need to show that

$$f_G(x,y) := \sum_{T \subseteq G} x^{\mathrm{int}(G)} y^{\mathrm{ext}(G)}$$

satisfies the deletion-contraction recurrence

$$(*) \quad f_G = f_{G \setminus e} + f_{G/e}$$

(& the initial conditions).

It is hard to check $(*)$ for any edge e. But it is easier to prove $(*)$ if we assume that e is the minimal edge in E. But it is already enough to check that $(*)$ holds for some edge e, in order to deduce that $f_G(x,y) = T_G(x,y)$ by induction on $|E|$. $\square$

Special Values of $T_G(x,y)$:

- $T_G(1,1)$ = # spanning trees of G (if G is connected)

- $\chi_G(t) = (-1)^{n-k} t^k \, T_G(1-t, 0)$

  k = # connected components of G.

- $T_G(2,0)$ = # acyclic orientations of G.

- $T_G(2,1)$ = # forests in G.

- $T_G(2,2) = 2^{|E|}$

- .... Many other graphical invariants are expressed as specializations of $T_G(x,y)$.

The tree inversion polynomial

Recall, $I_n(y) :=$

$$= \sum_{\substack{T \text{ spanning} \\ \text{tree of } K_{n+1}}} y^{\mathrm{inv}(T)}$$

Proposition

$$I_n(y) = T_{K_{n+1}}(1, y)$$

Proof. There is a way to order the set of edges of K_{n+1} s.t. $\mathrm{ext}(T) = \mathrm{inv}(T)$. $\square$

Also recall that $I_n(-1)$ is the number A_n of alternating permutations of size n.

So the value $T_G(1,-1)$ is a generalization of the number A_n to any graph G.

The Tutte polynomial appears in many different areas of math & physics, for example:

- Statistical Physics (Ising & Potts model)
- Knot theory (Jones & HOMFLY polynomials)
- etc.

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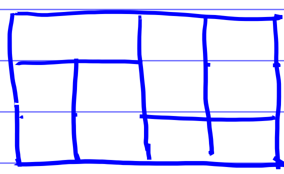
Domino Tilings

(Last week we discussed rhombus tilings.)

Def. A domino tiling is a way to subdivide some region on the plane (typically, an $m \times n$ rectangle) into dominos (1×2 or 2×1 rectangles).

Example $m = 3$, $n = 4$

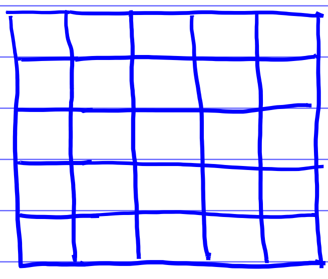


a domino tiling of 3×4 rectangle

Clearly, we can tile an $m \times n$ rectangle by dominos iff $m \cdot n$ is even.

What if both m & n are odd?

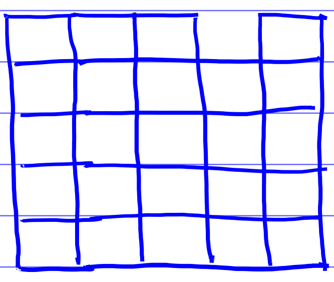
<u>Example</u> $m = n = 5$



5×5 square

We <u>cannot</u> subdivide the 5×5 square into domino, because it has the odd number 5^2 of boxes.

How about the region obtained by removing a single box, e.g.

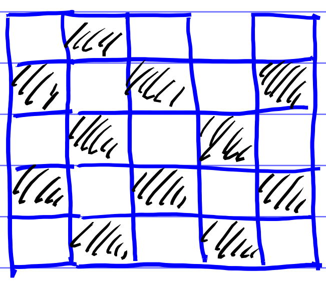


$5^2 - 1 = 24$ boxes

Can we tile this region by dominos?

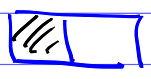
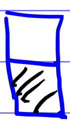
Answer : No

Let's color all boxes in black & white like a chessboard:



11 black boxes

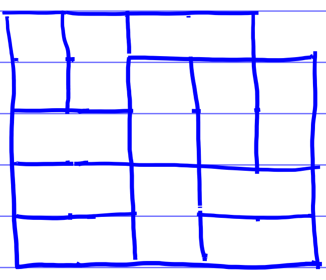
13 white boxes

But any domino ( ) should contain exactly one black box and exactly one white box.

So we <u>cannot</u> tile this region by dominos.

In order to have the same numbers of black & white boxes, the color of the removed box should be the same as the color of a corner box.

<u>Example</u>



We can tile the 5×5 square without a corner box by dominos

Can we find the number domino tilings?

Theorem (Kasteleyn 1961)

Assume that n is even,

domino tilings of an $m \times n$ rectangle equals

$$\prod_{k=1}^{n/2} \prod_{\ell=1}^{\lfloor m/2 \rfloor} \left(4\cos^2\frac{\pi k}{n+1} + 4\cos^2\frac{\pi \ell}{m+1}\right).$$

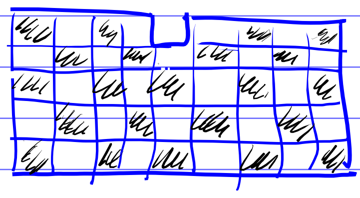
The proof is based on a clever way to express the permanent of a certain matrix as the determinant of another matrix.

Theorem (Temperley, 1974)

Suppose that m & n are both odd $m = 2k+1$, $n = 2\ell+1$.

Consider a region R obtained from a $m \times n$ rectangle by removing a single box b s.t.

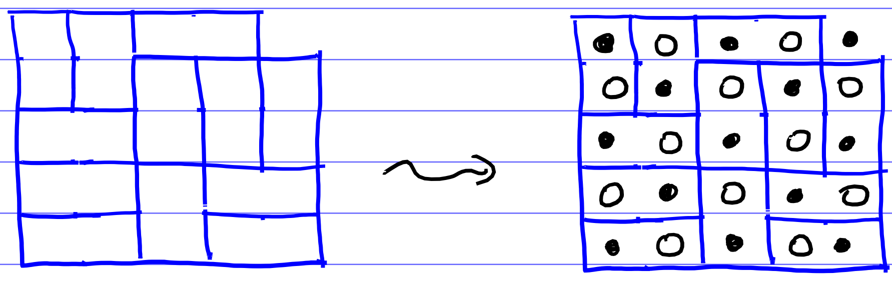
- b is on the boundary of the rectangle
- b has the same color as corners of the rectangle

$R =$ 

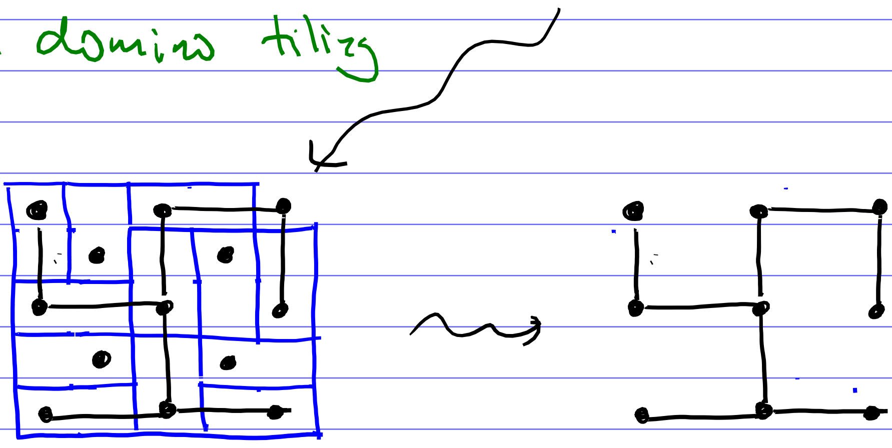
Then # domino tilings equals # spanning trees of the $k \times \ell$ grid graph.

Proof Let's construct a bijection between domino tilings & spanning trees.

Example



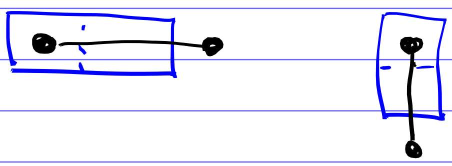
a domino tiling



Spanning tree of 3×3 grid graph

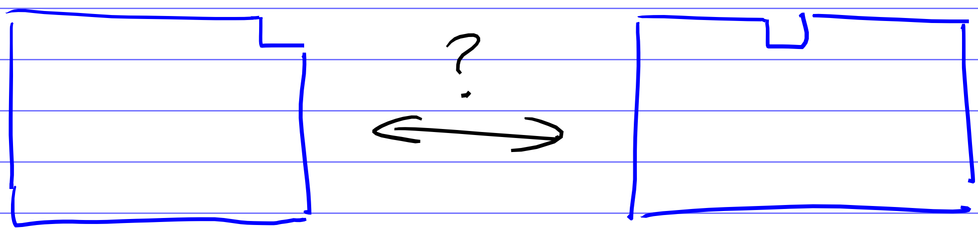
Rule for edges:

Connect the black dots like this:



Claim: this construction give a bijection between domino tilings & spanning trees.

Observation If we remove any other box b of the same color on the boundary of the $m\times n$ rectangle, we get the same number of domino tilings.



a bijection between domino tilings