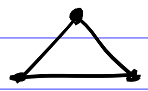
Chromatic polynomial

$G = (V, E)$ an undirected graph on a finite vertex set V.
(We'll usually assume $V = \{1, 2, \dots, n\}$.)

Definition.
For $k \in \mathbb{Z}_{>0}$, a function $c : V \to \{1, 2, \dots, k\}$ is called a <u>(proper) k-coloring</u> if $c(u) \neq c(v)$ for any edge $(u, v) \in E$.

Examples. (1) $G = K_3 =$ 

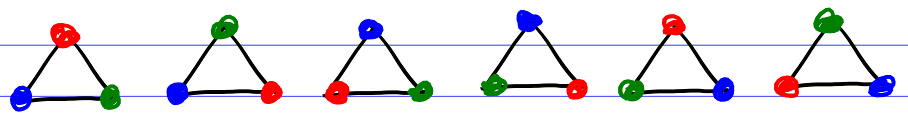
- no 1-colorings

  A graph G has a 1-coloring iff $E = \emptyset$.

- no 2-colorings

  A graph has a 2-coloring iff G is a bipartite graph.

- $\exists$ a 3-coloring
  (1 = •, 2 = •, 3 = •)

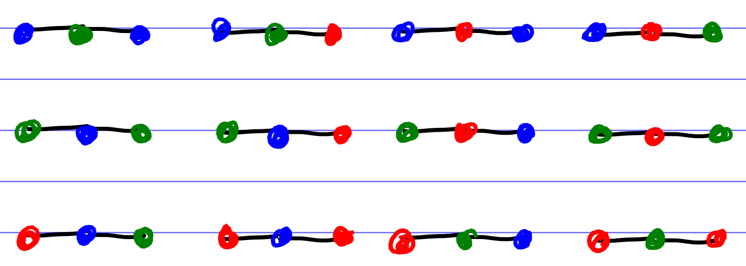


K_3 has 6 3-colorings.

(2) $G =$

- no 1-colorings
- G has 2 2-colorings:

- G has $3 \cdot 2 \cdot 2 = 12$ 3-colorings:



Lemma. There exists a unique polynomial $\chi_G(t)$ such that, for any positive integer k, $\chi_G(k)$ equals the number of k-colorings of G.

Moreover, $\chi_G(t)$ has integer coefficients.

Definition. $\chi_G(t)$ is called the **chromatic polynomial** of graph G.

The minimal number $k \in \mathbb{Z}_{>0}$ such that $\chi_G(k) \neq 0$ is called the **chromatic number**

Examples (1) $G = K_3$

$\#$ k-colorings $= k\cdot(k-1)\cdot(k-2)$

$\chi_{K_3}(t) = t\cdot(t-1)\cdot(t-2)$

the chromatic number is 3.

(2) $G =$ 

$\#$ k-colorings $= k\cdot(k-1)(k-1)$

$\chi(t) = t\cdot(t-1)(t-1)$

the chromatic number is 2.

Proof of Lemma. The uniqueness claim is clear. If two polynomials coincide at infinitely many points, then the polynomials are equal to each other.

The proof of existance is by induction on $|E|$ ← # of edges in G.

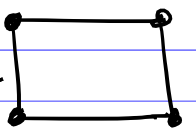
Deletion – Contraction

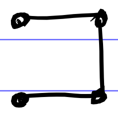
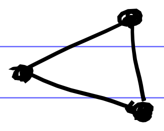
Assume that $E \neq \emptyset$, and pick one edge $e \in E$.

G

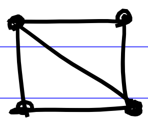
$G \setminus e$ — the graph with deleted edge e

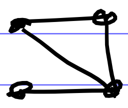
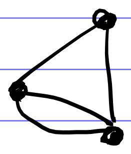
G/e — the graph with contracted edge e

Examples (1) $G =$ e 

$G \setminus e =$ , $G/e =$ 

deletion — contraction

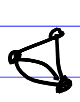
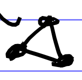
(2) $G =$ e 

$G \setminus e =$  $G/e =$ 

deletion — contraction

Notice that G/e might contain multiple edges even if G has no multiple edges.

However, for the chromatic polynomial $\chi_G(t)$ multiple edges don't matter.

For example, χ$(t) = \chi$(t).

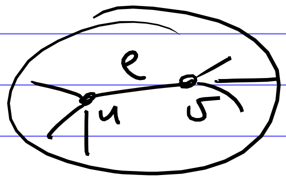
We can remove all multiple edges without affecting $\chi_G(t)$.

Deletion - Contraction recurrence for k- colorings:

$\#\{k\text{-colorings of } G\} =$

$= \#\{k\text{-colorings of } G \setminus e\}$

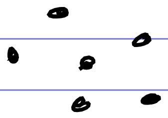
$- \#\{k\text{-colorings of } G/e\}.$

Indeed, $G =$ 

the k-colorings of $G \setminus e$ which are not k-colorings of G, are exactly the k-colorings $c: V \to \{1, 2, \dots, k\}$ such that $c(u) = c(v)$. They correspond to the k-colorings of G/e (contraction).

... back to the proof of lemma.

Induction on $|E|$.

<u>Base</u>. $E = \emptyset$ $\quad G =$ 

(the empty graph on n vertices)

$\#$ k-colorings equals k^n (a polynomial in k)

So $\chi_G(t) = t^n$.

Induction step $E \neq \emptyset$

$e \in E$.

By induction $\chi_{G \setminus e}(t)$ and $\chi_{G/e}(t)$ are polynomials.

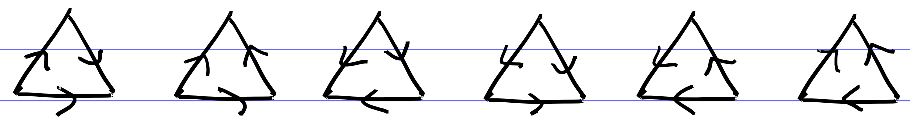
So $\chi_G(t) = \chi_{G \setminus e}(t) - \chi_{G/e}(t)$ is a polynomial.

Moreover, we prove by induction that $\chi_G(t)$ has integer coefficients. $\square$

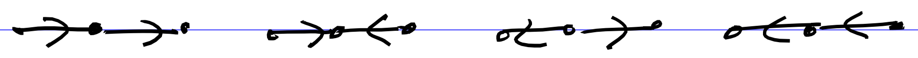
Acyclic orientations

Definition. An *acyclic orientation* of an (undirected) graph G is a way to direct its edges so that the resulting directed graph has no directed cycles.

Examples (1) $G = K_3$ has 6 acyclic orientations:

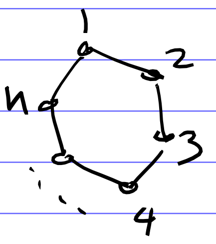


(2) $G =$ has 4 acyclic orient.

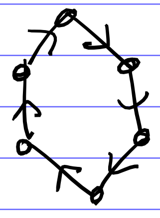
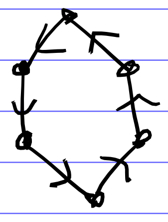


(All orientations are acyclic)

(3) More generally, a forest has $2^{|E|}$ acyclic orientations

(4) An n-cycle $G =$  has $2^n - 2$ acyclic orientations.

All orientations are acyclic, except the two orientations:

 and  .

Theorem (R. Stanley, 1973)

Let G be a graph on n vertices. Then the number of acyclic orientations of G, equals $(-1)^n \chi_G(-1)$.

Examples. $G = K_3$

$$\chi_{K_3}(t) = t\cdot(t-1)(t-2).$$

$$(-1)^3 \chi_{K_3}(-1) = -(-1)(-2)(-3)$$
$$= 6.$$

K_3 has 6 acyclic orientations.

(2) $G =$ 

$$\chi_G(t) = t\cdot(t-1)^2$$

$$(-1)^3 \chi_G(-1) = -(-1)(-2)^2 = 4$$

G has 4 acyclic orientations.

What k-colorings & acyclic orientations have in common?

Deletion - Contraction

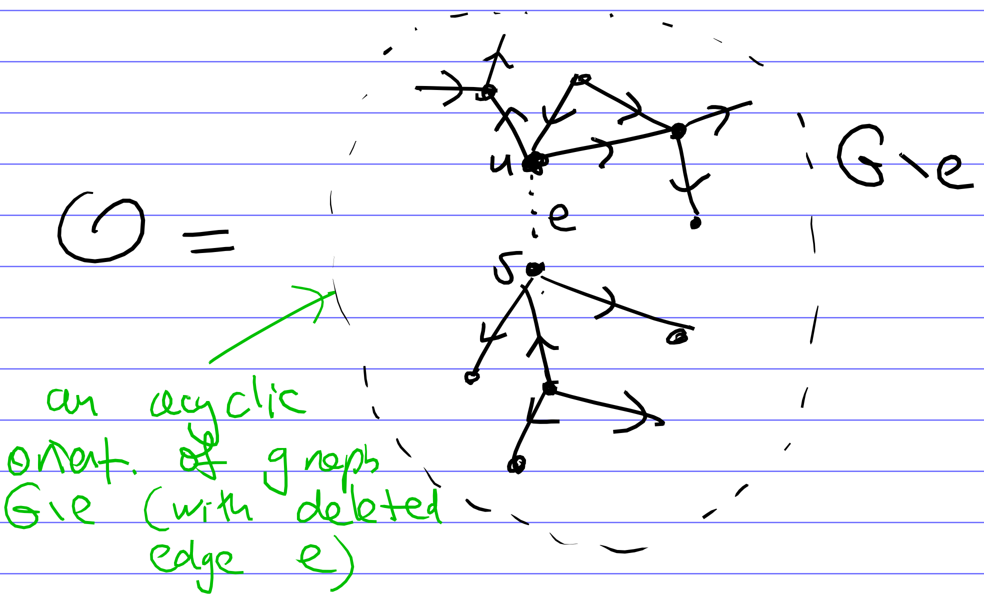
Let $AO(G) := \#$ acyclic orientations of G.

Lemma. For an edge e of G

$$AO(G) = AO(G \setminus e) + AO(G/e).$$

Proof. Let $\mathcal{O}$ be an acyclic orientation of $G \setminus e$. There can be 2 or 1 way to extend $\mathcal{O}$ to an acyclic orientation of G.

$\mathcal{O} =$



I. If $\mathcal{O}$ does not have a directed path from u to v, or from v to u, then we can orient the edge e in either of the 2 ways. (2 ways to extend an acyclic orientation)

II. If G has a directed path between u and v, say, a path from u to v, then there is only one way to extend the acyclic orientation. Namely, the edge e should be oriented from u to v.

Now observe that, if we contract the edge e, in case (I) we get an acyclic orientation of G/e, but in case (II) we get an orientation of G/e with a directed cycle.

So we deduce that

$$AO(G) = AO(G \setminus e) + AO(G/e)$$

$\square$

Notice that we have slightly different deletion-contraction recurrences:

$$\chi_G(t) = \chi_{G\setminus e}(t) - \chi_{G/e}(t)$$

$$AO(G) = AO(G\setminus e) + AO(G/e)$$

The factor $(-1)^n$ takes care of this.

<u>Proof of Stanley's theorem</u>

$$AO(G) = (-1)^n \chi_G(-1).$$

Induction on $|E|$.

<u>Base</u> $E = \emptyset$

$AO(G) = 1, \quad \chi_G(t) = t^n$

$(-1)^n \cdot (-1)^n = 1$ ✓

<u>Induction Step</u> $e \in E$

$$AO(G) = AO(G\setminus e) + AO(G/e)$$

$$= (-1)^n \chi_{G\setminus e}(-1) + (-1)^{n-1} \chi_{G/e}(-1)$$

$$= (-1)^n \left(\chi_{G\setminus e}(-1) - \chi_{G/e}(-1)\right)$$

G/e has $n-1$ vertices

$$= (-1)^n \chi_G(-1),$$ as needed.

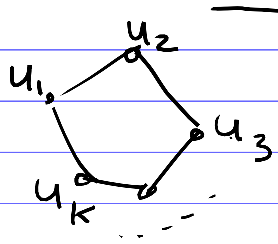
$\square$

How to calculate $\chi_G(t)$?

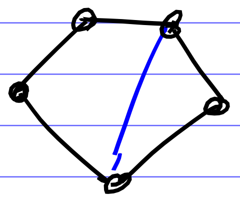
There is a nice class of graphs, for which $\chi_G(t)$ is given by a simple product formula.

Chordal Graphs

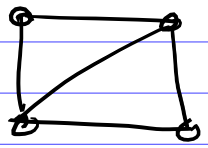
Definition A simple graph G is called chordal if any cycle

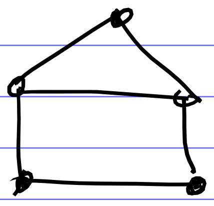


in G of length $r \geq 4$ has a chord, i.e. pair of vertices v_i & v_j ($j \neq i \pm 1 \mod r$) connected by an edge.



a chord in a 5-cycle

Examples.  is chordal



is not chordal (the 4-cycle does not have a chord)

Definition A **perfect elimination ordering** of a graph G is an ordering $v_1, \ldots, v_n$ of all its vertices such that

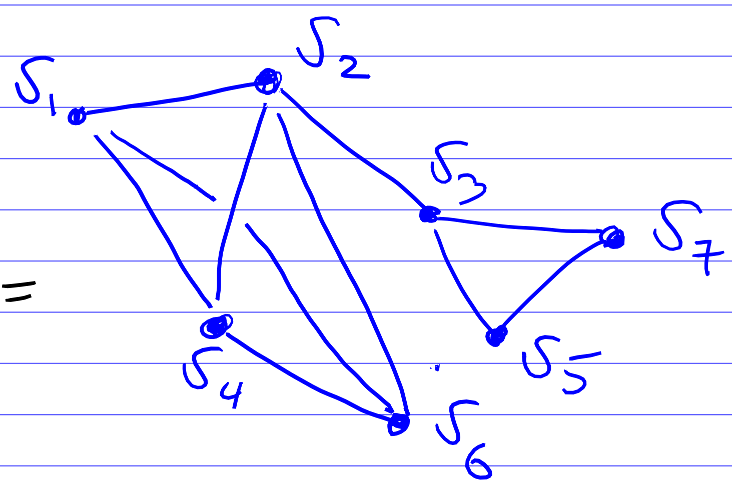
$$\forall\ i = 2, 3, \ldots, n$$

the subset of vertices

$$\{ v_j \mid j < i,\ (v_j, v_i) \text{ is an edge in } G \}$$

forms a clique, i.e., a complete subgraph in G.

Example

$G =$ 

a perfect elimination ordering of vertice in G

Theorem (Fulkerson-Gross 1965)

A simple graph G is chordal iff it has a perfect elimination ordering of vertices.

Remark One direction ($\Leftarrow$) is easy.

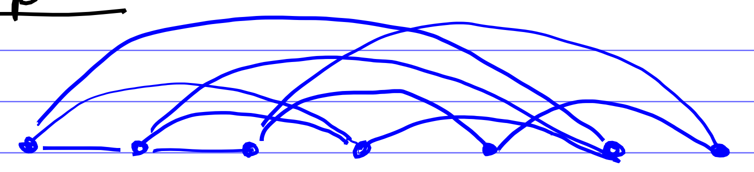
For a perfect elimination ordering $v_1, \ldots, v_n$ of G define the numbers

$a_1, a_2, \ldots, a_n \in \mathbb{Z}_{\geq 0}$

$$a_i := \#\{ j < i \mid (v_j, v_i) \text{ is an edge} \}$$

(We always have $a_1 = 0$.)

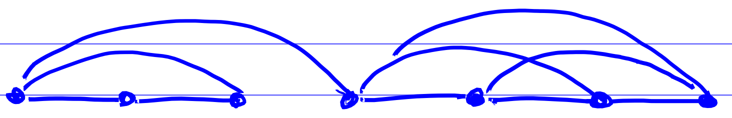
Example

$G =$ 

$v_1 \quad v_2 \quad v_3 \quad v_4 \quad v_5 \quad v_6 \quad v_7$

a_1	a_2	a_3	a_4	a_5	a_6	a_7
0	1	1	2	1	3	2

Another perfect elimination ordering of the same graph

$G =$ 

v_1'	v_2'	v_3'	v_4'	v_5'	v_6'	v_7'
v_3	v_5	v_7	v_2	v_6	v_4	v_1

a_1'	a_2'	a_3'	a_4'	a_5'	a_6'	a_7'
0	1	2	1	1	2	3

We obtain two <u>different</u> sequences:

$$(a_1, \ldots, a_7) = (0, 1, 1, 2, 1, 3, 2)$$

$$(a_1', \ldots, a_7') = (0, 1, 2, 1, 1, 2, 3)$$

Notice that these two sequences a permutations of each other.

Why?

Theorem Let G be a chordal graph. Let $(a_1, \ldots, a_n)$ be the sequence obtained from any perfect elimination ordering of G.

Then

$$\chi_G(t) = (t-a_1)(t-a_2)\cdots(t-a_n).$$

Corollary. # acyclic orientations of a chordal graph equals

$$(a_1+1)(a_2+1)\cdots(a_n+1).$$

Example For the above graph G, we have

$$\chi_G(t) = t\cdot(t-1)^3(t-2)^2(t-3),$$

and $AO(G) = 1\cdot 2^3\cdot 3^2\cdot 4.$

Proof of Thm. Let color the vertices $v_1, v_2, \ldots, v_n$ of G one by one starting from v_1

k-colorings of G:

$k = k - a_1$ ways to color v_1

$k - a_2$ ways to color v_2

$k - a_3$ ways to color v_3, etc.

Notice that, at each step, there are exactly a_i colors which we cannot use to color v_i. These are the colors of the preceding vertices v_j, $j<i$ connected to v_i.

Since these a_i vertices form a clique in G, they all have different colors.

So # k-coloring of G is

$(k-a_1)(k-a_2)\cdots(k-a_n)$

$$\Rightarrow \chi_G(t) = \prod_{i=1}^{n}(t-a_i).$$

$\square$

There are several invariants*

of graphs that satisfy
(a version) of deletion–
contraction recurrence:

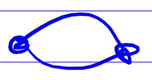
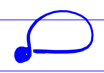
- # acyclic orientations of G
- chromatic polynomial $\chi_G(t)$
- # spanning trees $ST(G)$:

$$ST(G) = ST(G \setminus e) + ST(G/e)$$

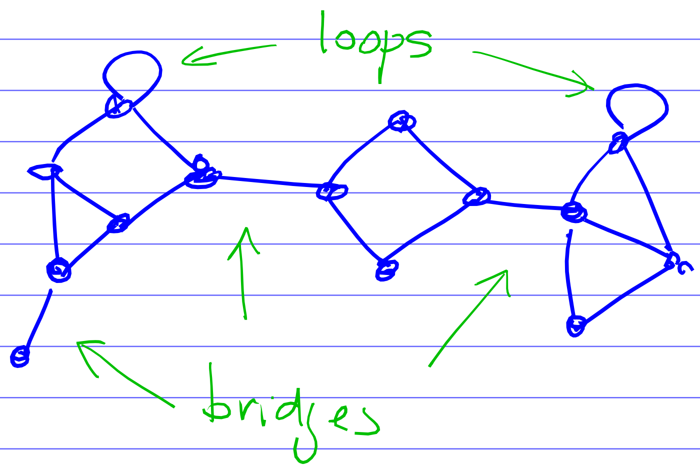
* Here the word "invariant" means that all these numbers and polynomials don't depend on a choice of ordering of the vertices of G.

Is there the most general graphical invariant that satisfies the deletion–contraction recurrence?

The Tutte polynomial $T_G(x,y)$

G - an undirected graph
(we allow multiple edges 
and loops .)

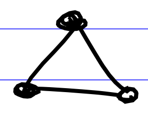
An edge e in G is called a bridge (a.k.a. isthmus) if $G \setminus e$ has more connected component than G.

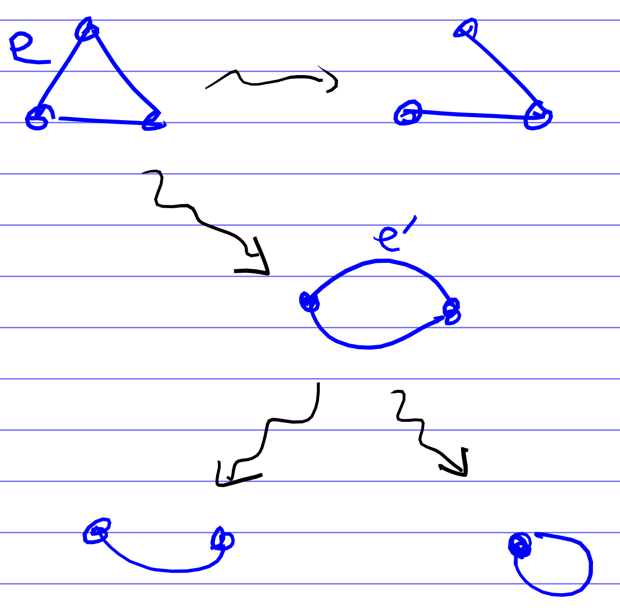
Example 

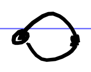
Theorem (Tutte) There exists a unique polynomial $T_G = T_G(x,y)$, with positive integer coefficients, such that

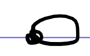
- For any edge e in G, which is not a loop nor a bridge, we have
$$T_G = T_{G \setminus e} + T_{G/e}$$

- If G has a bridges and b loops (and no other edges), then
$$T_G = x^a y^b.$$

Example $G = K_3 =$ 



$T_{K_3} = T$ $+\, T$

$= x^2 + (T$ $+\, T$$)$

$= x^2 + x + y.$

All other graphical invariants we discussed today ($\chi_G(t)$, $AO(G)$, $ST(G)$) are special values of the Tutte polynomial

Non-recursive formula for $T_G(x,y)$: internal and external activities

Fix a total ordering of the set E of edges of G.

Let $T \subset G$ be a spanning tree of G.

Definition (1) An edge $e \in T$ is called internally active if $\nexists$ smaller edge $e' < e$ s.t. $(T \setminus \{e\}) \cup \{e'\}$ is a spanning tree.

(2) An edge $f \in G \setminus T$ is called externally active if $\nexists$ smaller edge f' s.t. $(T \cup \{f\}) \setminus \{f'\}$ is a spanning tree.

Let

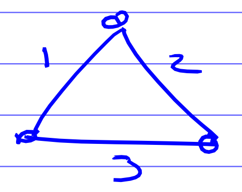
$$\mathrm{int}(T) := \#\left\{\begin{matrix}\text{internally active} \\ \text{edges w.r.t. tree } T\end{matrix}\right\}$$

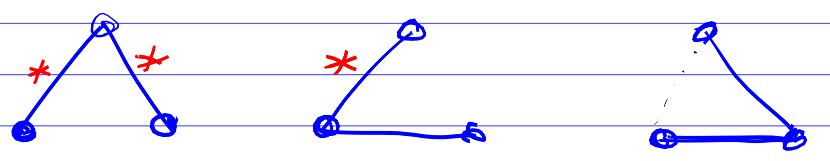
$$\mathrm{ext}(T) := \#\left\{\begin{matrix}\text{externally active} \\ \text{edges w.r.t. } T\end{matrix}\right\}$$

Theorem

$$T_G(x,y) = \sum_{\substack{T \text{ is a} \\ \text{spanning tree} \\ \text{of } G}} x^{\mathrm{int}(T)}\, y^{\mathrm{ext}(T)}$$

Remark Clearly $\mathrm{int}(T)$ and $\mathrm{ext}(T)$ depend on a choice of total order of edges in G. But the polynomial $T_G(x,y)$ does not depend on this choice.

Example $G =$ 



int $= 2$, ext $= 0$ int $= 1$, ext $= 0$ int $= 0$, ext $= 1$

$$T_{K_3} = x^2 + x + y.$$

Special Values of $T_G(x,y)$:

- $T_G(1,1)$ = # spanning trees
- $\chi_G(t) = (-1)^{n-k} t^k T_G(1-t, 0)$

  k = # connected components of G.
- $T_G(2,0)$ = # acyclic orientations of G.
- $T_G(2,1)$ = # forests in G.
- $T_G(2,2) = 2^{|E|}$