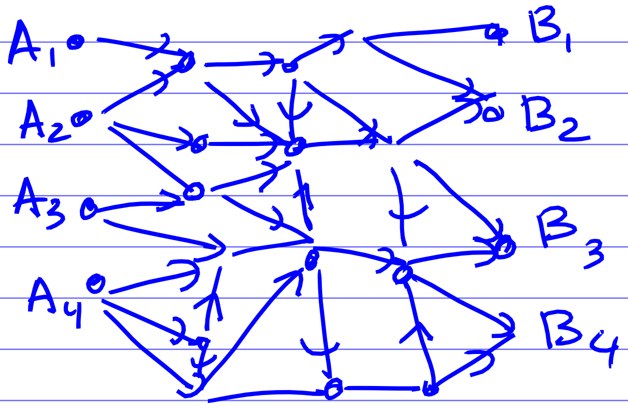
Last time: Lindström - Gessel - Viennot Lemma

G - planar directed acyclic weighted graph with vertices $A_1, \dots, A_n$ on the "left side" & $B_1, \dots, B_n$ on the right side.

$G =$ 

$C = (c_{ij})$ $n \times n$ matrix

$$c_{ij} = \sum_{P: A_i \to B_j}' \text{weight}(P)$$

Then $$\sum_{\substack{(P_1, \dots, P_n) \\ P_i : A_i \to B_i, i = 1, \dots, n \\ P_i \cap P_j = \emptyset \ \forall i \neq j}}' \prod_{i=1}^{n} \text{weight}(P_i) = \det(C)$$

collection of non-crossing paths

Plane Partitions

Fix $m, n, k \geq 1$

A *plane partition* (of shape $m \times n$) is a filling of the $m \times n$ rectangle by non-negative integers $\in \{0, 1, \dots, k\}$ (equiv. a $m \times n$ matrix) such that the entries weakly decrease in rows & columns.

Example $m = 3,\ n = 4,\ k = 2$

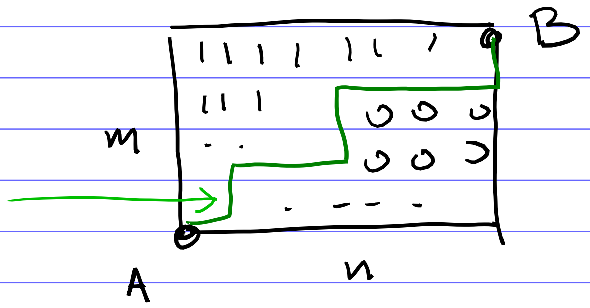
$$\begin{array}{c} \geq \\ \begin{array}{|c|c|c|c|} \hline 2 & 2 & 1 & 1 \\ \hline 2 & 1 & 1 & 0 \\ \hline 2 & 1 & 1 & 0 \\ \hline \end{array} \end{array}$$

a plane partition of shape 3×4 with entries $\in \{0, 1, 2\}$

How to find the number of plane partition (for given m, n, k)?

Case $k = 1$.

the lattice path that separates 1's & 0's

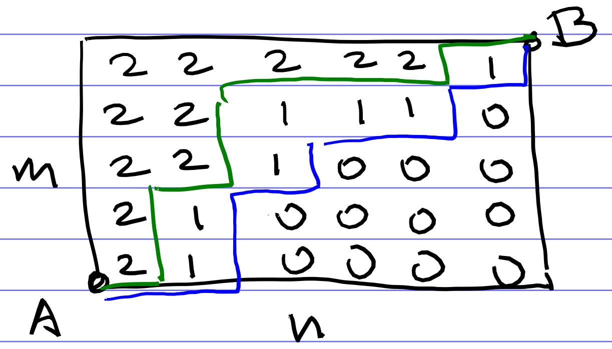


$\#$ plane partitions $=$

$= \#$ lattice paths from A to B

$$= \binom{m+n}{m}$$

Case $k=2$

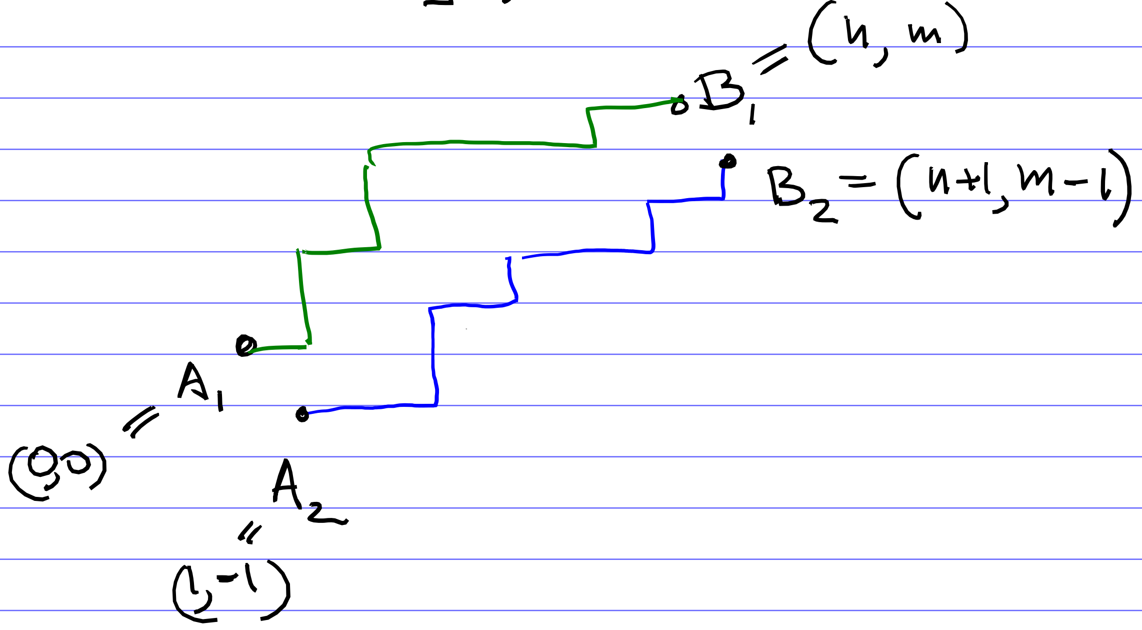


two lattices paths between 2's & 1's and between 1's & 0's.

plane partitions =

= # pairs of lattice paths from A to B that cannot cross each other (in the strict sense) but can "touch" each other.

Let's shift the second path by 1 step in the direction "↘".

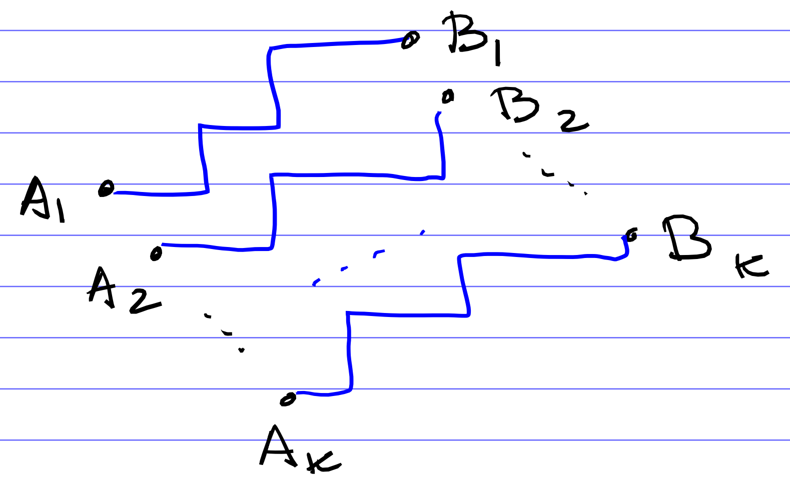


Now we get a pair of non-crossing paths from A_1 & A_2 to B_1 & B_2, which we can count using Lidström-Gessel-Viennot Lemma:

$$\begin{vmatrix} \binom{m+n}{m} & \binom{m+n}{m-1} \\ \binom{m+n}{m+1} & \binom{m+n}{m} \end{vmatrix}$$

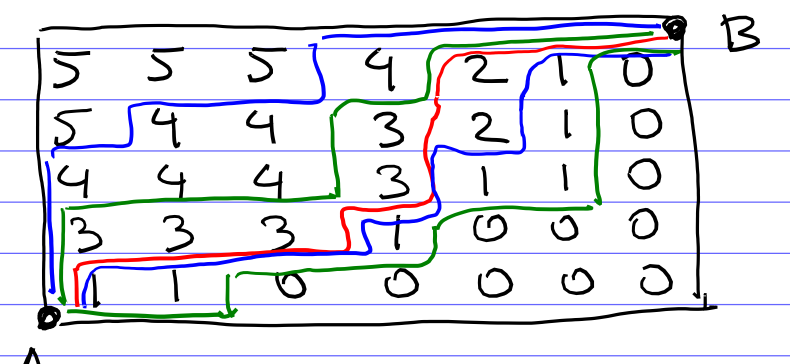
We can do a similar construction for any k:
Shift the second path by $(1,-1)$, shift the third path by $(2,-2)$, etc.

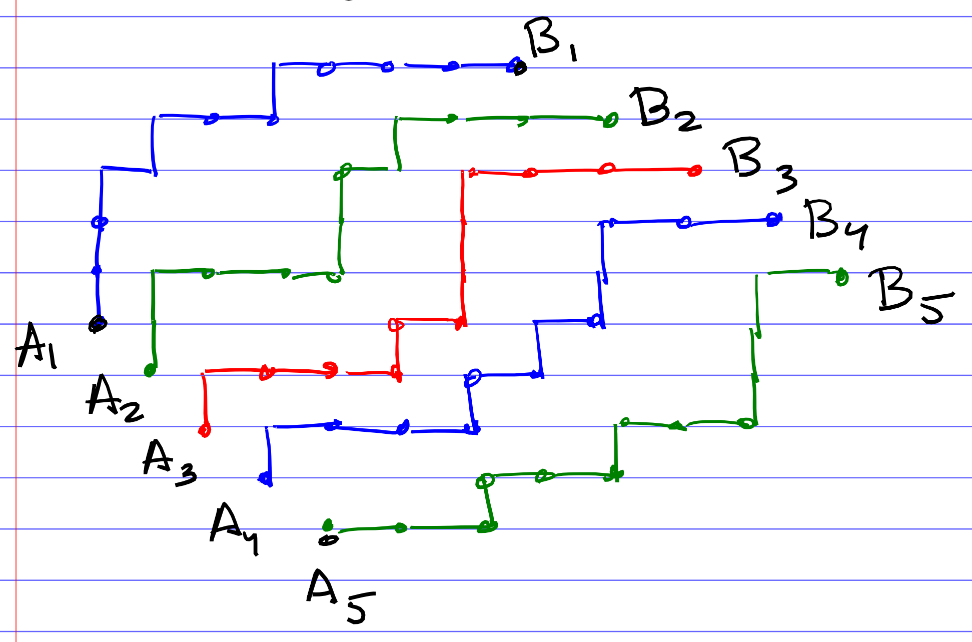
For general k, # plane partitions with entries $\in \{0,1,\dots,k\}$ equals # k-tuples of non-crossing paths connecting A_i's with B_i's



$A_i = (i-1, -i+1)$, $B_i = (n+i-1, m-i+1)$
for $i = 1, 2, \dots, k$

<u>Example</u> $(m, n, k) = (5, 7, 5)$





Linström-Gessel-Viennot Lemma implies:

<u>Theorem</u> # plane partitions of shape $m \times n$ with entries $\in \{0,1,\dots,k\}$ equals the determinant of the $k \times k$ matrix

$$\det \begin{bmatrix} \binom{m+n}{m} & \binom{m+n}{m-1} & \cdots & \binom{m+n}{m-k+1} \\ \binom{m+n}{m+1} & \binom{m+n}{m} & \cdots & \binom{m+n}{m-k+2} \\ \cdots & \cdots & \cdots & \cdots \\ \binom{m+n}{m+k-1} & \binom{m+n}{m+k-2} & \cdots & \binom{m+n}{m} \end{bmatrix}.$$

Actually there is a more explicit formula for # plane partitions.

<u>Theorem</u> (A. MacMahon 1896 proved in "Combinatory Analysis" 1916)

plane partitions of shape $m \times n$ with entries $\in \{0, 1, \ldots, k\}$ equals

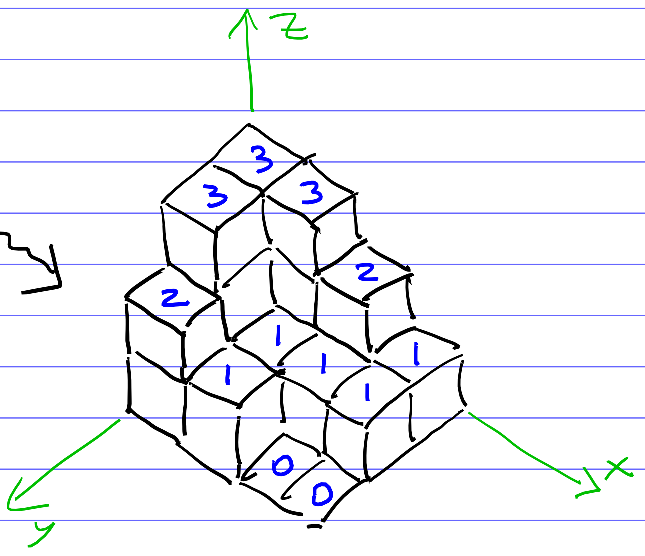
$$\prod_{a=1}^{m} \prod_{b=1}^{n} \prod_{c=1}^{k} \frac{a+b+c-1}{a+b+c-2}.$$

<u>Remark</u> Notice that the resulting formula is symmetric in m, n, k. This symmetry becomes clear if we represent plane partitions as "3-dimensional Young diagrams".

<u>Example</u> $(m, n, k) = (3, 4, 3)$

3	3	2	1
3	1	1	1
2	1	0	0

a plane partition



If a box of the plane partition is filled with ℓ, then put ℓ $1 \times 1 \times 1$ cubes above it.

We get a "3-dim Young diagram" that should fit inside the $m \times n \times k$ box.

It is now clear that # such diagrams should be symmetric in m, n, k.

q-analog (also due to MacMahon)

For a plane partition $P = (p_{ij})$, let $|P| := \sum_{i,j} p_{ij}$

Theorem (MacMahon)

$$\sum_{\substack{P \text{ plane partition} \\ \text{of shape } m\times n \\ \text{with entries} \\ p_{ij}\in\{0,1,\dots,k\}}} q^{|P|} = \prod_{a=1}^{m}\prod_{b=1}^{n}\prod_{c=1}^{k} \frac{[a+b+c-1]_q}{[a+b+c-2]_q},$$

where $[\ell]_q := 1+q+q^2+\dots+q^{\ell-1}$.

Example $m=n=k=2$

$$\begin{array}{cccccccc}
\begin{bmatrix}0&0\\0&0\end{bmatrix} & \begin{bmatrix}1&0\\0&0\end{bmatrix} & \begin{bmatrix}2&0\\0&0\end{bmatrix} & \begin{bmatrix}1&1\\0&0\end{bmatrix} & \begin{bmatrix}1&0\\1&0\end{bmatrix} & \begin{bmatrix}2&1\\0&0\end{bmatrix} & \begin{bmatrix}2&0\\1&0\end{bmatrix} & \begin{bmatrix}1&1\\1&0\end{bmatrix} \\
1 & q & q^2 & q^2 & q^2 & q^3 & q^3 & q^3
\end{array}$$

$$\begin{array}{ccccccc}
\begin{bmatrix}2&2\\0&0\end{bmatrix} & \begin{bmatrix}2&0\\2&0\end{bmatrix} & \begin{bmatrix}2&1\\1&0\end{bmatrix} & \begin{bmatrix}1&1\\1&1\end{bmatrix} & \begin{bmatrix}2&2\\1&0\end{bmatrix} & \begin{bmatrix}2&1\\2&0\end{bmatrix} & \begin{bmatrix}2&1\\1&1\end{bmatrix} \\
q^4 & q^4 & q^4 & q^4 & q^5 & q^5 & q^5
\end{array}$$

$$\begin{array}{ccccc}
\begin{bmatrix}2&2\\2&0\end{bmatrix} & \begin{bmatrix}2&2\\1&1\end{bmatrix} & \begin{bmatrix}2&1\\2&1\end{bmatrix} & \begin{bmatrix}2&2\\2&1\end{bmatrix} & \begin{bmatrix}2&2\\2&2\end{bmatrix} \\
q^6 & q^6 & q^6 & q^7 & q^8
\end{array}$$

$$1+q+3q^2+3q^3+4q^4+3q^5+3q^6+q^7+q^8$$

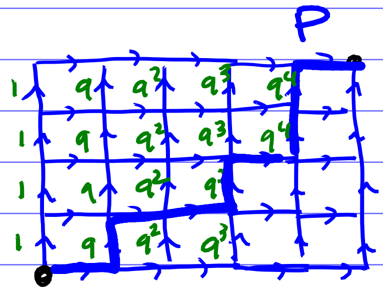
$$= \frac{[2]_q[3]_q[3]_q[3]_q[4]_q[4]_q[4]_q[5]_q}{[1]_q[2]_q[2]_q[2]_q[3]_q[3]_q[3]_q[4]_q}$$

$$= (1+q^2)^2(1+q+q^2+q^3+q^4).$$

Is there a determinantal formula for the q-analog of the # of plane partitions?

Consider the same graph (the square grid) with weights of horizontal edges $= 1$ and weight of a vertical edge $(i,j) \to (i,j+1)$ equal q^i

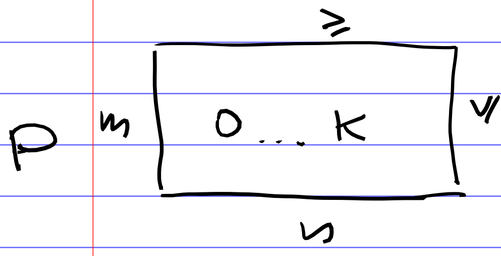
Example



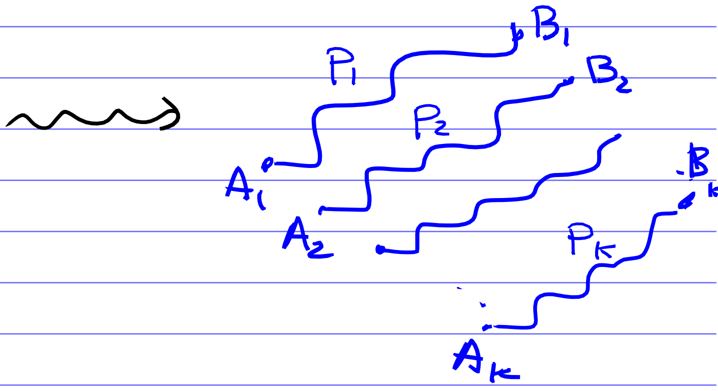
$$\text{weight}(P) = q \cdot q^3 \cdot q^4 \cdot q^4 = q^{12}$$
$$= q^{\#\{\text{boxes above path } P\}}$$

$$\sum_{P:(0,0)\to(n,m)} \text{weight}(P) = \begin{bmatrix} m+n \\ n \end{bmatrix}_q$$

the Gaussian q-binomial coeff.



a plane partition



k non-crossing lattice paths

When we shift the paths, their weights change:

weight (P_1) stays the same

weight (P_2) multiplied by q^n

weight (P_3) multiplied by q^{2n}

etc.

So

$$q^{|P|} = q^{m(1+2+\dots+k-1)} \prod_{i=1}^{k} \text{weight}(P_i)$$

$C =$

$$\begin{bmatrix} \begin{bmatrix} m+n \\ m \end{bmatrix}_q & \begin{bmatrix} m+n \\ m-1 \end{bmatrix}_q & \cdots & \begin{bmatrix} m+n \\ m-k+1 \end{bmatrix}_q \\ q^{m+1}\begin{bmatrix} m+n \\ m+1 \end{bmatrix}_q & q^{n}\begin{bmatrix} m+n \\ m \end{bmatrix}_q & \cdots & q^{m-k+2}\begin{bmatrix} m+n \\ m-k+2 \end{bmatrix}_q \\ \cdots & \cdots & & \cdots \\ q^{(k-1)(m+k-1)}\begin{bmatrix} m+n \\ m+k-1 \end{bmatrix}_q & & \cdots & q^{(k-1)m}\begin{bmatrix} m+n \\ m \end{bmatrix}_q \end{bmatrix}$$

Linström-Gessel-Viennot Lemma $\Rightarrow$

$$\sum_{\substack{P \text{ plane} \\ \text{partitions} \\ \text{of shape } m\times n \\ \text{with entries } \in \{0,1,..,k\}}} q^{|P|} = \det(C),$$

where $c_{ij} = q^{(i-1)(i-j)} \begin{bmatrix} m+n \\ m+i-j \end{bmatrix}_q$

for $i,j \in \{1,2,..,k\}$.

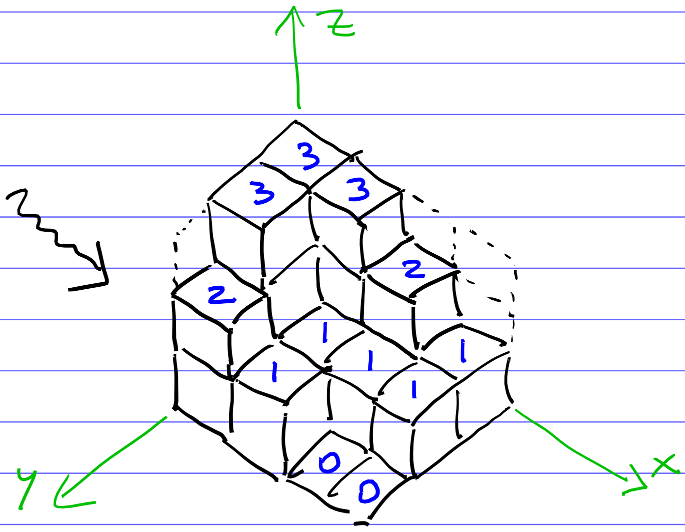
Plane Partitions can be bijectively identified with several other types of combinatorial objects:

- Rhombus tilings
- Perfect matchings
- Pseudo-line arrangements

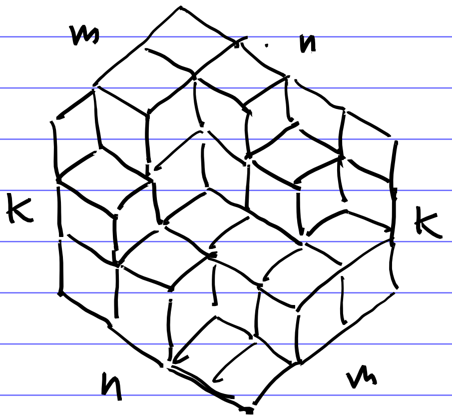
Example

$$\begin{array}{c|cccc|} \multicolumn{5}{c}{n} \\ \hline & 3 & 3 & 2 & 1 \\ m & 3 & 1 & 1 & 1 \\ & 2 & 1 & 0 & 0 \\ \hline \end{array}$$

a plane partition

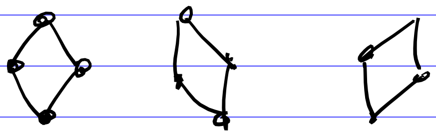


"3D Young diagram"

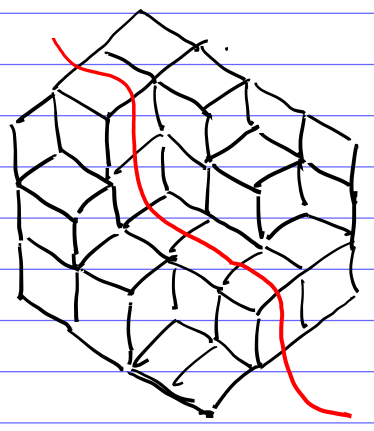


a rhombus tiling

a tiling of the hexagon with sides of lengths m, n, k, m, n, k (clockwise) by rhombuses:

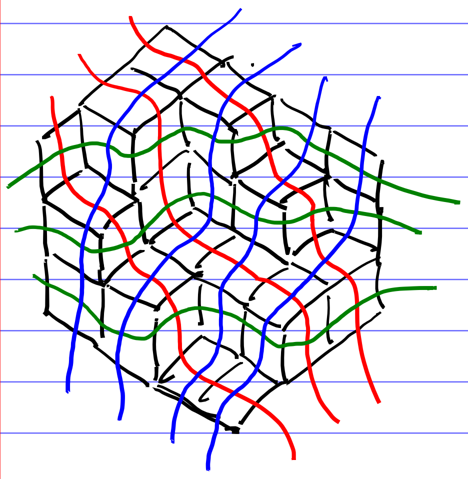


Pseudo-line Arrangements

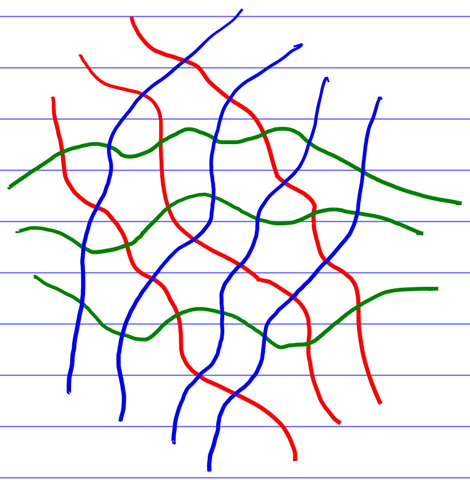


a pseudoline

- start at one one boundary edge of the hexagon
- go to the opposite edge of a little rhombus
- continue until you reach the opposite side of the hexagon.



$\rightsquigarrow$



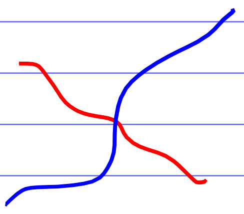
draw all pseudolines

a pseudoline arrangement

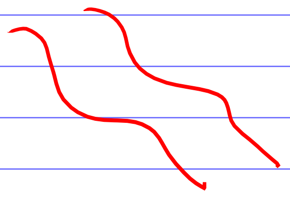
this is exactly the planar dual graph of the rhombus tiling

A **pseudoline arrangement** is a drawing on the plane (considered up to homotopy) that has pseudolines of 3 colors (m red, n blue, and k green) such that

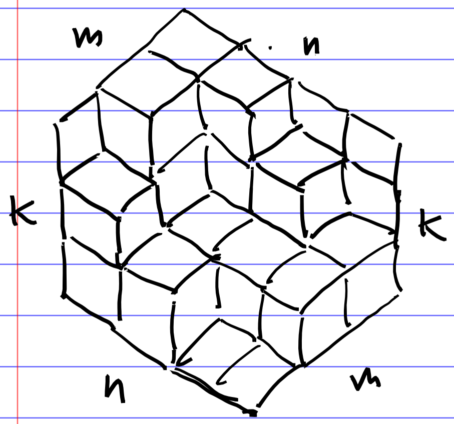
- any two pseudolines of different colors intersect exactly once



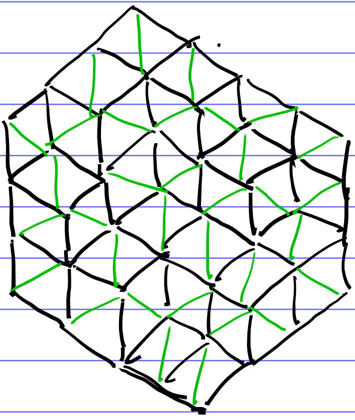
- any two pseudolines of the same color don't intersect



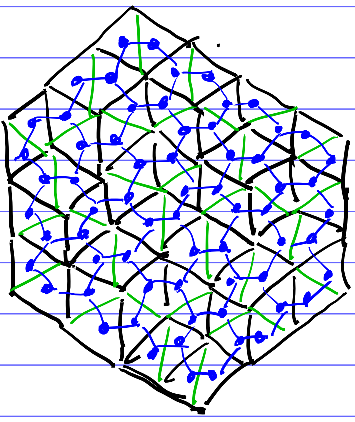
Perfect Matchings



a rhombus tiling

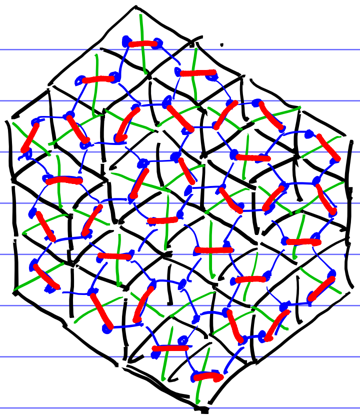


subdivide each rhombus into two triangles

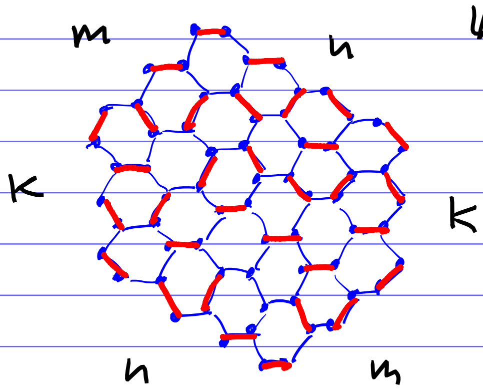


draw the plane dual graph:

- put a vertex • inside each triangle
- connect adjacent vertices by edges



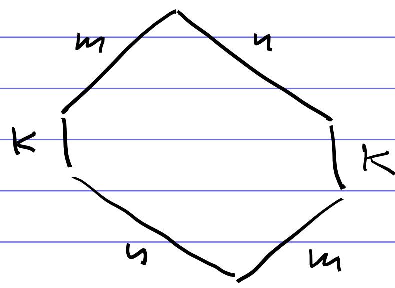
mark the edges of the dual graph that intersect the edges subdividing the rhombuses (green edges)



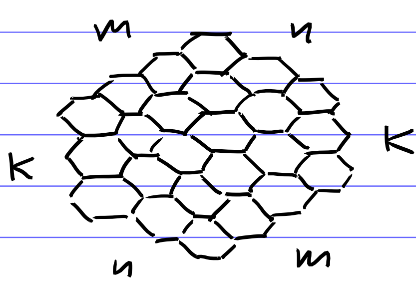
we obtain a <u>perfect</u> <u>matching</u> of the honeycomb graph

<u>Theorem</u> The following objects are in bijection with each other:

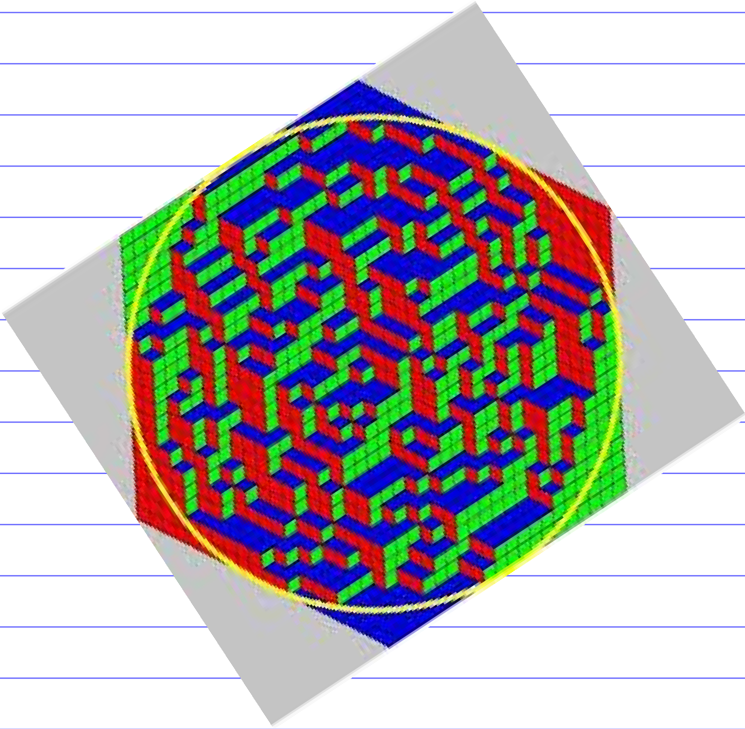
- <u>plane partitions</u> of shape $m \times n$ with entries $\{0,1,\dots,k\}$
- <u>rhombus tilings</u> of the hexagon



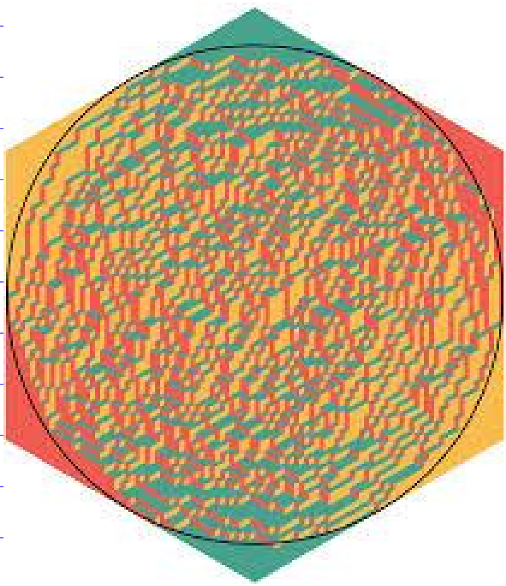
- <u>pseudo-line arrangements</u> with m red, n blue, and k green pseudo-lines,
- <u>perfect matchings</u> of the honeycomb graph:



Here is a random rhombus tiling:



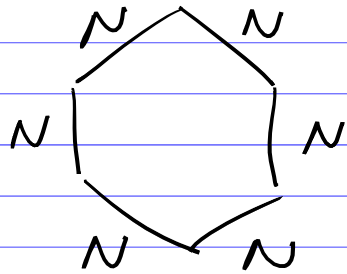
Another random rhombus tiling for larger $m=n=k$.



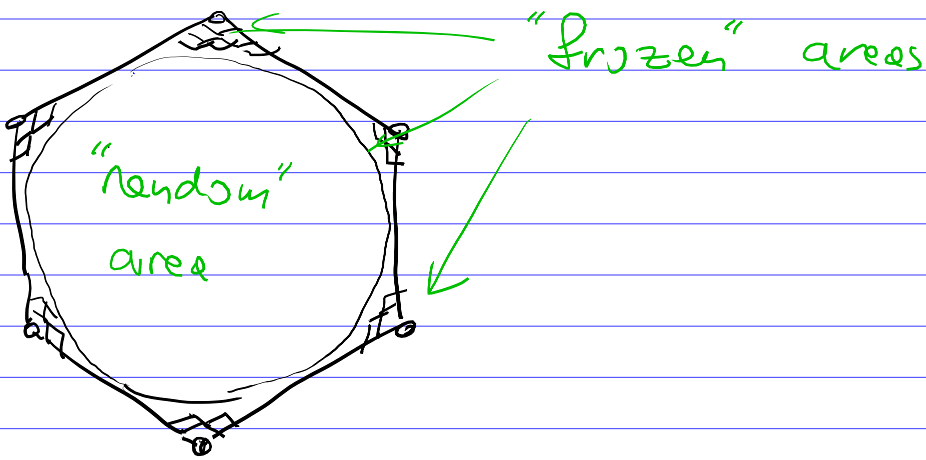
As you can see, the tiling "looks random" inside the circle and "frozen" outside the circle

"Arctic Circle" Phenomenon

For $N \to \infty$, random rhombus tilings of the large hexagon



are "frozen" outside the circle & random inside the circle:



Another Application of Lindström - Gessel - Viennot Lemma

Consider the "Catalan matrix"

$$\begin{bmatrix} C_0 & C_1 & C_2 & \cdots & C_{n-1} \\ C_1 & C_2 & C_3 & \cdots & C_n \\ C_2 & C_3 & C_4 & \cdots & C_{n+1} \\ \cdots & & & & \\ C_{n-1} & C_n & C_{n+1} & \cdots & C_{2(n-1)} \end{bmatrix}$$

$$C_i = \frac{1}{i+1}\binom{2i}{i}$$

Example $n=1$ $\quad |C_0| = 1$

$n=2$
$$\begin{vmatrix} C_0 & C_1 \\ C_1 & C_2 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix}$$

$$= 1\cdot 5 - 2\cdot 2 = 1$$

$n=3$
$$\begin{vmatrix} C_0 & C_1 & C_2 \\ C_1 & C_2 & C_3 \\ C_2 & C_3 & C_4 \end{vmatrix} =$$

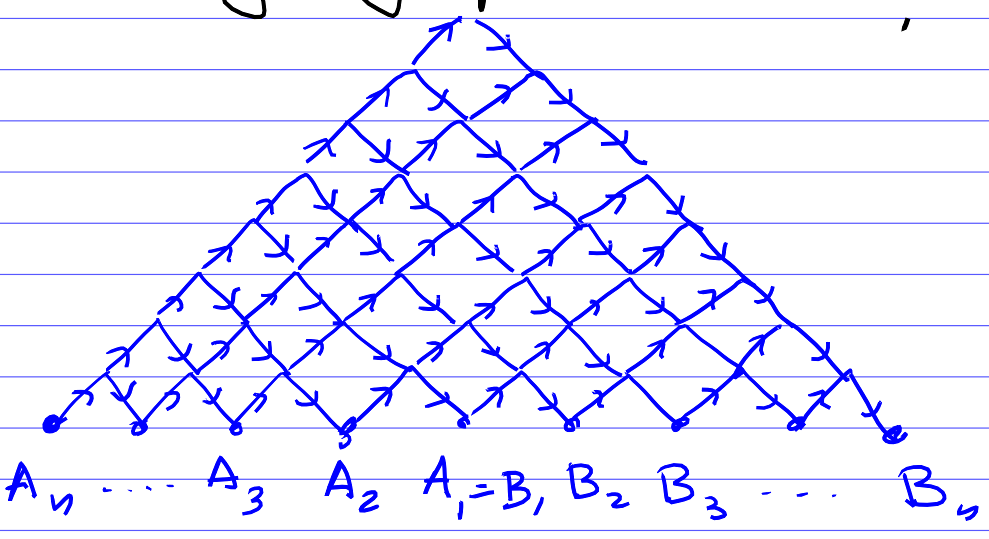
$$= \begin{vmatrix} 1 & 2 & 5 \\ 2 & 5 & 14 \\ 5 & 14 & 42 \end{vmatrix} = \ldots = 1$$

Proposition The determinant of the $n \times n$ matrix

$$\begin{bmatrix} C_0 & C_1 & \cdots & C_{n-1} \\ C_1 & C_2 & \cdots & C_n \\ \cdots & & & \\ C_{n-1} & C_n & & C_{2(n-1)} \end{bmatrix}$$

equals 1.

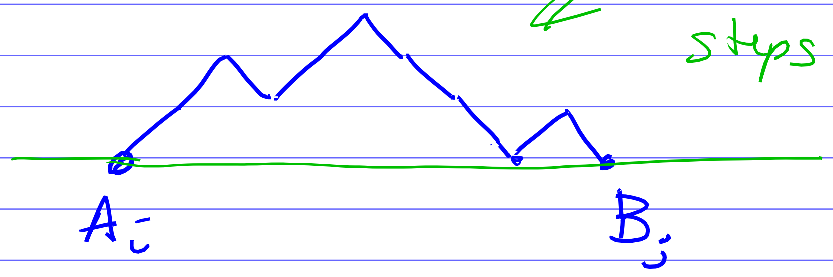
Proof. Consider the following graph



directed paths from A_i to B_j equals the Catalan number

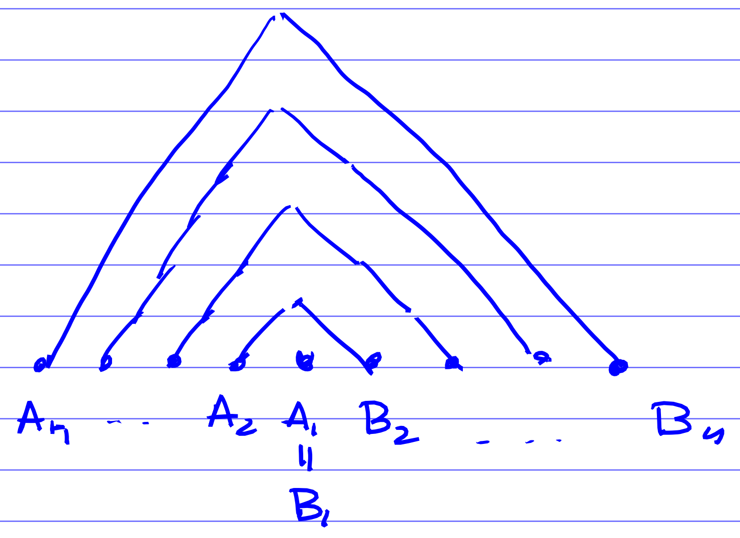
C_{i+j-2}.

Dyck paths with $2(i+j-2)$ steps



So the determinant of this $n \times n$ matrix equals # ways to connect A_i's with B_i's by non-crossing paths.

But there is only 1 way to do this.



So the determinant equals 1.