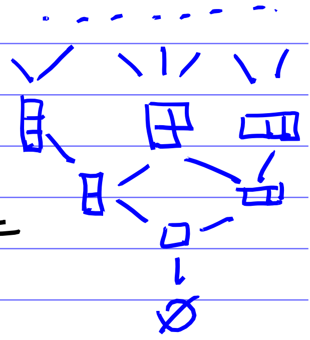
<u>last time</u>: differential posets

$$[D,U] = I, \text{ where}$$

the commutator of D & U $\rightarrow$ $[D,U] := DU - UD.$

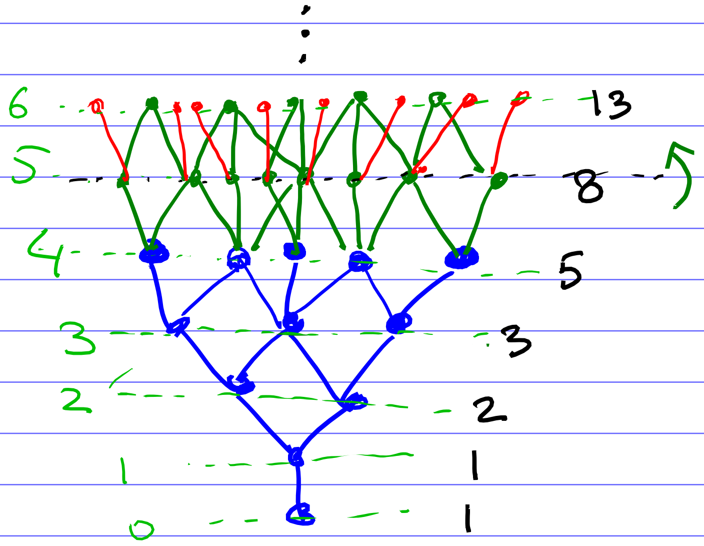
<u>examples of diff. posets</u>:

- <u>Young's lattice</u> $\mathbb{Y}$ =



- <u>Fibonacci lattice</u>

$\mathbb{F}$ =



<u>Theorem</u> The n^{th} rank number r_n of $\mathbb{F}$ equals the $(n+1)^{st}$ Fibonacci number F_{n+1}.

n	0	1	2	3	4	5	6	
F_{n+1}	1	1	2	3	5	8	13	...

$$F_1 = F_2 = 1, \qquad F_{n+1} = F_n + F_{n-1}.$$

not partitions

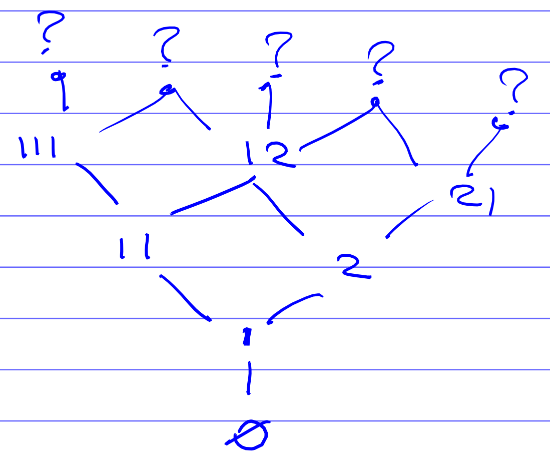
Recall, that F_{n+1} equals # compositions of n with all parts equal 1 or 2.

n	0	1	2	3	4	...
F_{n+1}	1	1	2	3	5	...
compositions with parts 1 & 2	$\varnothing$	1	2, 11	21, 12, 111	22, 211, 121, 112, 1111	

Indeed, it is easy to get the Fibonacci recurrence for # such compositions $c=(c_1,\dots,c_\ell)$ by removing the last part $c_\ell \in \{1,2\}$ of a composition:

- # comp. of n s.t. $c_\ell = 1$ $=$ # comp. of $n-1$
- # comp of n s.t. $c_\ell = 2$ $=$ # comp of $n-2$.

$\mathbb{F} =$



<u>Questions.</u> How to label the elements of $\mathbb{F}$ by compositions with all parts 1 & 2?

How to describe the covering relation in terms of such compositions?

How to see that $\mathbb{F}$ is a lattice?

<u>Theorem</u> Like Young's lattice, the Fibonacci lattice $\mathbb{F}$ satisfies:

$$\sum_{\substack{\lambda \in \mathbb{F} \\ \text{rank}(\lambda)=n}} \left(f_\lambda(\mathbb{F})\right)^2 = n!$$

saturated chains in $\mathbb{F}$ from $\hat{0}$ to λ.

What about more general paths in the Hasse diagram of a differential poset P?

Let w be any word in the letters "U" & "D" with exactly n U's and exactly n D's. Let w^{rev} be its reversed word.

<u>Example</u>, $n=4$

$$w = UUDUUDDD$$

$$w^{rev} = DDDUUDUU.$$

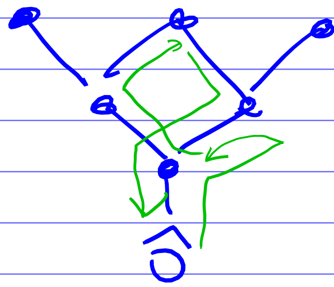
We can view w^{rev} as a product of "up" & "down" operators and define

$f_w(P)$ is the number N such that $w^{rev}(\hat{0}) = N\hat{0}$.

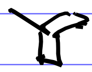
Combinatorially,

$f_w(P) := \#$ paths in the Hasse diagram of P that start and end at $\hat{0}$ with the pattern of up & down steps given by the word w.

Example $\quad w = UUDUUDDD$

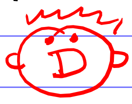
$P =$ 

$$f_w(P) = 2 \cdot 6 = 12.$$

Def. For $P =$  such paths (with an arbitrary pattern of "up" & "down" steps) are called

oscillating tableau.

So we are counting # oscillating tableaux that start & end at $\hat{0}$ and have a given pattern of up & down steps.

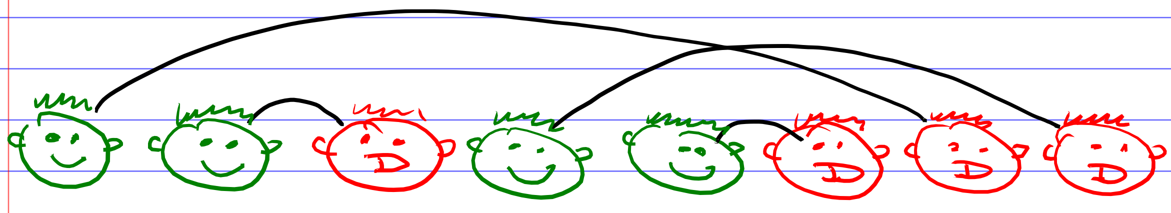
Using the same construction with hopping particles  and anti-particles  we can deduce that

Theorem $f_w(P)$ equals # ways to match all U's with all D's so that each U is matched with a D to the right of it (in the word w).

Example. $w = UUDUUDDD$

$$DDDUUDUU\,(\hat{0}) = N\,\hat{0},$$

w^{rev} where N = # matchings like this

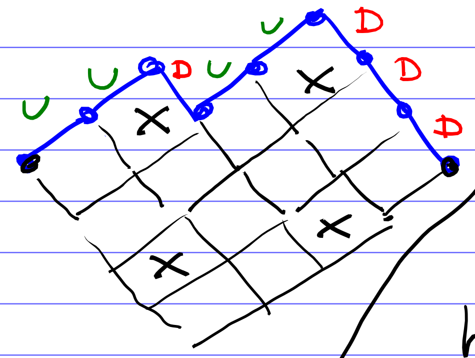


U U D U U D D D

Notice that f_w is non-zero only if w is a Dyck word.

Let's convert w to a Dyck path.

Example. $w = UUDUUDDD$



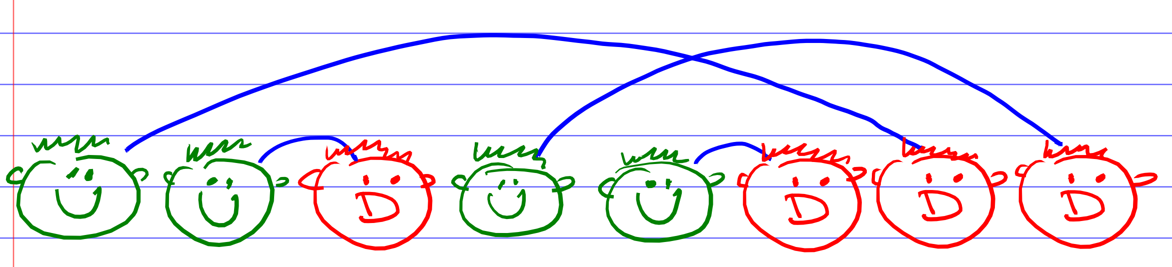
λ $=(4,4,3,3)$

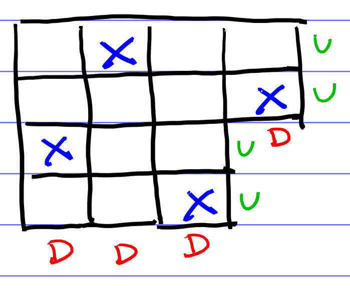
Let λ be the Young diagr. (rotated by 135°) below this Dyck path.

Rook Placements

Notice that a matching of U's & D's corresponds to a placement of n non-attacking rooks on the "chessboard" $\varkappa$

Example $w = UUDUUDDD$



$\varkappa =$ 

Theorem. $f_w =$

$=$ # perfect matchings (as above)

$=$ # rook placements (with n rooks) on the board of shape $\varkappa = (\varkappa_1 \geq \varkappa_2 \geq \dots \geq \varkappa_n)$.

$= \varkappa_n \cdot (\varkappa_{n-1} - 1)(\varkappa_{n-2} - 2) \cdots (\varkappa_1 - n + 1)$.

We can count # such rook placements, as follows. We should have exactly 1 rook in each row of shape $\varkappa$ and in each column of the shape $\varkappa$.

- There are $\varkappa_n$ ways to place a rook in the <u>last</u> row of $\varkappa$.
- There are $\varkappa_{n-1} - 1$ ways to place a rook in the second to the last row of $\varkappa$ so that it does not attack the 1st rook.
- $\varkappa_{n-2} - 2$ ways to place a rook in the next row, etc.

In total, we get.

Theorem # placements of n non-attacking rooks on the board of shape $\varkappa = (\varkappa_1, \dots, \varkappa_n) \subseteq n \times n$ equals

$$\boxed{\varkappa_n (\varkappa_{n-1} - 1)(\varkappa_{n-2} - 2) \cdots (\varkappa_1 - n + 1)}$$

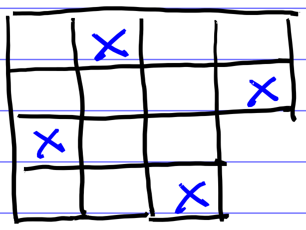
assuming that all factors > 0. (otherwise # rook placements is zero.)

We can also place the rooks by columns & get the expression:

$$\varkappa_n' \cdot (\varkappa_{n-1}' - 2)(\varkappa_{n-2}' - 2) \cdots (\varkappa_1' - n + 1).$$

where $\varkappa' = (\varkappa_1', \dots, \varkappa_n')$ is the conjugate partition to $\varkappa$.

Example $\varkappa = (4,4,3,3)$



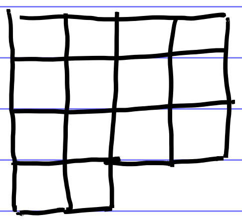
4
4
3
3

$\#$ rook placements $=$

$\underline{3 \cdot 2 \cdot 2 \cdot 1} = 12$

For the conjugate shape

$\varkappa' = (4,4,4,2)$

$=$ 

4
4
4
2

we also get $2 \cdot 3 \cdot 2 \cdot 1 = 12$

Exercise Prove combinatorially that, for a Young diagram $\varkappa \subseteq n \times n$, we have,

$$\varkappa_n \cdot (\varkappa_{n-1} - 1)(\varkappa_{n-2} - 2) \cdots (\varkappa_1 - n + 1)$$

$$= \varkappa'_n \cdot (\varkappa'_{n-1} - 1)(\varkappa'_{n-2} - 2) \cdots (\varkappa'_1 - n + 1)$$

assuming that all the factors are positive.

Question: Will this remain true if we drop the last assumption?

We can also easily deduce that f_w is given by this product using differential operators (as in Proof #2 from last lecture)

<u>Example</u> $w = UUDUUDDD$

f_w equals $DDDUUDUU(1)$

where $U: g(x) \longmapsto x\,g(x)$

$D: g(x) \longmapsto g'(x)$.

We get

$$f_w = \left(x^2\left(x^2\right)'\right)'''$$

$$= \left(x^2 \cdot 2x\right)'''$$

$$= 2 \cdot 3 \cdot 2 \cdot 1 = 12.$$

Back to the Robinson-Schensted correspondence...

Let's now assume that P is Young's lattice $\mathbb{Y}$.

(P, Q) a pair of SYT's of the same shape $\lambda \vdash n$

corresponds to a path on (the Hasse diagram of) $\mathbb{Y}$ starting & ending at $\emptyset$ with n "up" steps followed by n "down" steps.

Let's construct a bijection between such paths on $\mathbb{Y}$ and permutations in S_n using the relation:

$$[D, U] = I.$$

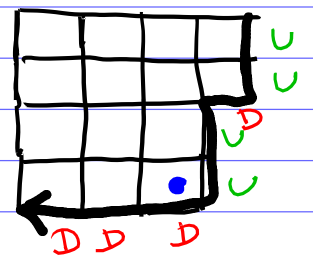
More generally, we will construct a <u>bijection</u> $\varphi_\varkappa$ between all paths in $\mathbb{Y}$ (i.e. oscillating tableaux) from $\emptyset$ to $\emptyset$ with a given sequence of up & down steps & rook placements on the associated shape $\varkappa \subset n \times n$.

We'll construct $\varphi_\varkappa$ using induction on $|\varkappa|$.

<u>Base</u> $\varkappa = \emptyset$. ✓

Assume by induction that we have already constructed all bijections $\varphi_{\tilde{\varkappa}}$ for $\tilde{\varkappa}$ with $|\tilde{\varkappa}| < |\varkappa|$.

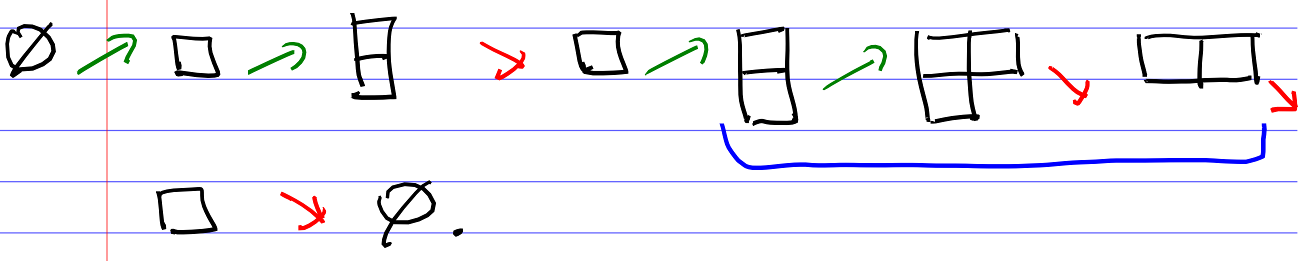
<u>Example.</u> $\varkappa = (4,4,3,3)$

$\varkappa =$ 

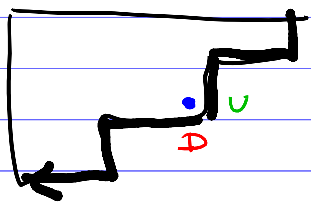
We are looking on paths in $\mathbb{Y}$ of the following form:

Up - Up - Down - Up - Up - Down - Down - Down

For example,



Let's find an inner corner

$\varkappa =$ 

in shape $\varkappa$

It corresponds to some up step followed by a Down step in the oscillating tableau T

$$T = \left(\emptyset \cdots \lambda \nearrow \mu \searrow \nu \cdots \emptyset\right)$$

Consider the 3 cases:

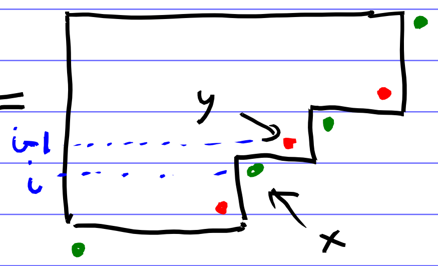
(I) $\lambda \neq \nu$

Define $\tilde{T}$ to be the unique osc. tableau obtained from T by replacing μ with $\tilde{\mu}$ s.t.

$$\tilde{T} = \left(\cdots \lambda \searrow \tilde{\mu} \nearrow \nu \cdots\right)$$

(II) $\lambda = \nu$ and μ is obtained from λ by adding a box $x \notin 1^{st}$ row of λ.

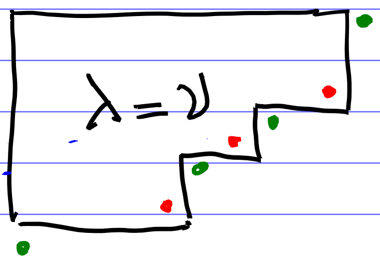
Let $\tilde{\mu}$ be the Young diagr. obtained from λ by removing a box y in the inner corner of λ that goes <u>right before</u> the outer corner x.

$\lambda = \nu =$ 

i.e. if $x \in i^{th}$ row then $x \in (i-1)^{st}$ row

Let $\tilde{T} = \left(\cdots \lambda \searrow \tilde{\mu} \nearrow \nu \cdots\right)$

(III) $\lambda = \nu$ & μ is obtained from λ by adding a box in 1^{st} row of λ



this is the "special" outer corner of λ.

Let $\tilde{T} = \left(\cdots \lambda \cdots\right)$

remove μ from T.

In cases (I) & (II)

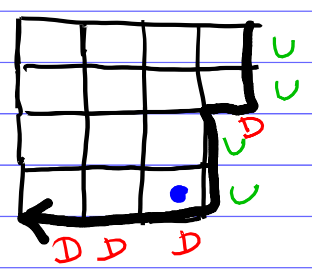
Let $\tilde{\varkappa}$ obtained from $\varkappa$ by removing the marked corner.

Define $\varphi_{\varkappa}(T) = \varphi_{\tilde{\varkappa}}(\tilde{T})$.

In case (III), let $\tilde{\tilde{\varkappa}}$ obtain from $\varkappa$ by removing the whole row & the whole column containg the marked corner.

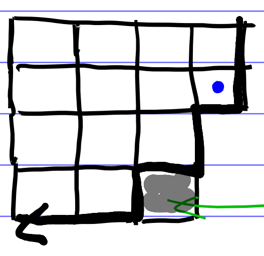
Define $\varphi_{\varkappa}(T)$ as the rook placement $\varphi_{\tilde{\tilde{\varkappa}}}(\tilde{\tilde{T}})$ together with one extra rook in the marked corner of $\varkappa$.

Example.

$\mathcal{X} =$ 

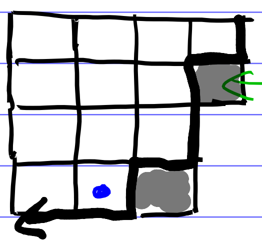
$T =$

$$\Big(\emptyset \nearrow \square \nearrow \genfrac{}{}{0pt}{}{\square}{\square} \searrow \square \nearrow \underbrace{\genfrac{}{}{0pt}{}{\square}{\square} \nearrow \boxplus \searrow \square\!\square \searrow}_{\text{case } (\mathrm{I})} \square \searrow \emptyset\Big).$$



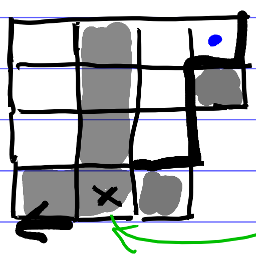
no rook in this box

$$\Big(\emptyset \nearrow \underbrace{\square \nearrow \genfrac{}{}{0pt}{}{\square}{\square} \searrow \square}_{\text{case } (\mathrm{II})} \nearrow \genfrac{}{}{0pt}{}{\square}{\square} \searrow \square \nearrow \square\!\square\!\square \searrow \square \searrow \emptyset\Big).$$



no rook at this box

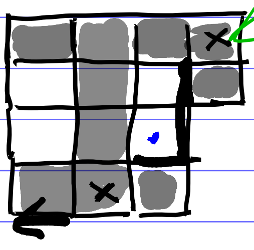
$$\Big(\emptyset \nearrow \square \searrow \emptyset \nearrow \square \nearrow \genfrac{}{}{0pt}{}{\square}{\square} \searrow \underbrace{\square \nearrow \square\!\square\!\square \searrow \square}_{\text{case } (\mathrm{III})} \searrow \emptyset\Big).$$



place a rook in this box.

$$\Big(\underbrace{\emptyset \nearrow \square \searrow \emptyset}_{\text{case } \mathrm{III}} \nearrow \square \nearrow \genfrac{}{}{0pt}{}{\square}{\square} \searrow \square \searrow \emptyset\Big)$$

place a rook in this box



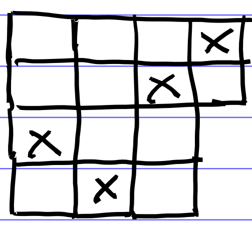
$$\Big(\emptyset \nearrow \underbrace{\square \nearrow \genfrac{}{}{0pt}{}{\square}{\square} \searrow \square}_{\text{case } \mathrm{II}} \searrow \emptyset\Big)$$

no rook in marked box

$$\underbrace{\emptyset \nearrow \square \searrow \emptyset}_{\text{case } \mathrm{III}} \underbrace{\nearrow \square \searrow \emptyset}_{\text{case } \mathrm{III}}$$

two more rooks.

The final rook placement is

$\varphi_{\mathcal{X}}(T) =$ 

Theorem. This construction gives a bijection between oscillating tableau & rook placements.

Proof. Not hard to see by induction.

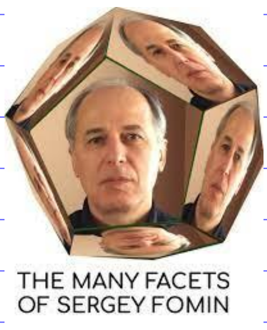
Theorem For $\varkappa = n \times n$ this correspondence coincides with the Robinson-Schensted correspondence:

$$\{(P,Q)\} \xleftrightarrow{RSK} S_n$$

Remark

Schensted's insertion algorithm is given by one particular way to remove corners from $\varkappa$.

Each Schensted's insertion step breaks into severall "corner removal" steps.



The above description of the Robinson-Schensted correspondence is related to Fomin's growth diagrams, 1994.

One benefit of this construction (vs. Schensted's insertion algorithm) is that its symmetry under reversal of the oscillating tableau is obvious from the construction.

Corollary

If $T \mapsto$ rook placement on shape $\varkappa$,

then $T^{rev} \mapsto$ the conjugate rook placement on shape $\varkappa'$.

In particular,

~~If~~ $(P, Q) \overset{RSK}{\longleftrightarrow} w \in S_n$

then $(Q, P) \overset{RSK}{\longleftrightarrow} w^{-1} \in S_n$

the inverse permutation corresponds to the conjugate rook placement.

$$w = \begin{array}{|c|c|c|c|} \hline & \times & & \\ \hline & & & \times \\ \hline \times & & & \\ \hline & & \times & \\ \hline \end{array} \rightsquigarrow \quad w^{-1} = \begin{array}{|c|c|c|c|} \hline & & \times & \\ \hline \times & & & \\ \hline & & & \times \\ \hline & \times & & \\ \hline \end{array}$$

Corollary.

without squares

$$\sum_{\lambda \vdash n}{}' f_\lambda = \#\left\{ w \in S_n \text{ s.t. } w = w^{-1} \right\}$$

Permutations $w \in S_n$ s.t. $w = w^{-1}$ are called called involutions.

An equivalent condition is: each cycle of w has size 2 or 1.

Exercise. Find an explicit formula for $\sum_{\lambda \vdash n} f_\lambda =$

$= \#\{ w \in S_n \mid w = w^{-1} \}$.

(The formula will involve a summation of some closed expression.)

Corollary. # all oscillating tableaux starting & ending at $\varnothing$ with $2n$ steps.

(i.e. # arbitrary paths from $\varnothing$ to $\varnothing$ with $2n$ steps) equals

perfect matchings in K_{2n}

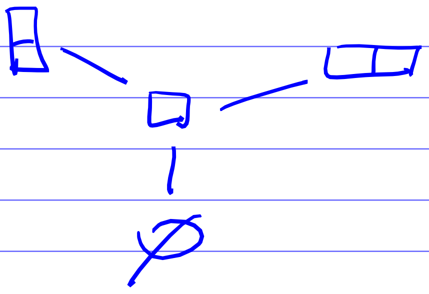
$= \#$ fixed-point-free involution is S_{2n}

$= (2n-1)!! := (2n-1)(2n-3)\cdots 1$

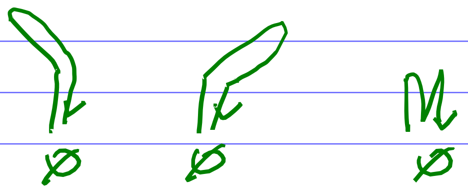
(problem 15 from PSet 1)

the product of all odd numbers between 1 and $2n-1$

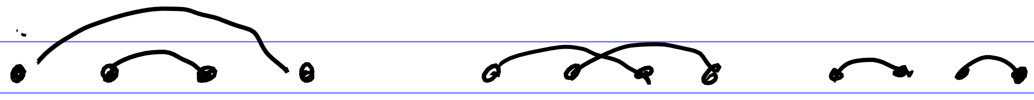
Example $n=2$



3 oscillating tableaux:



3 perfect matchings in K_4



Exercise. Construct a bijection between such oscillating tableaux & perfect matchings.

Exercise Prove the following theorem.

Theorem. # all oscillating tableaux (i.e. # paths in $\mathbb{Y}$) that

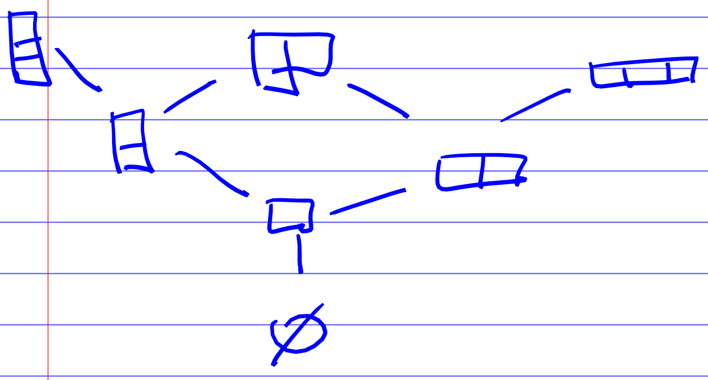
- start at $\varnothing$
- end at any Young diagr.
- have n steps

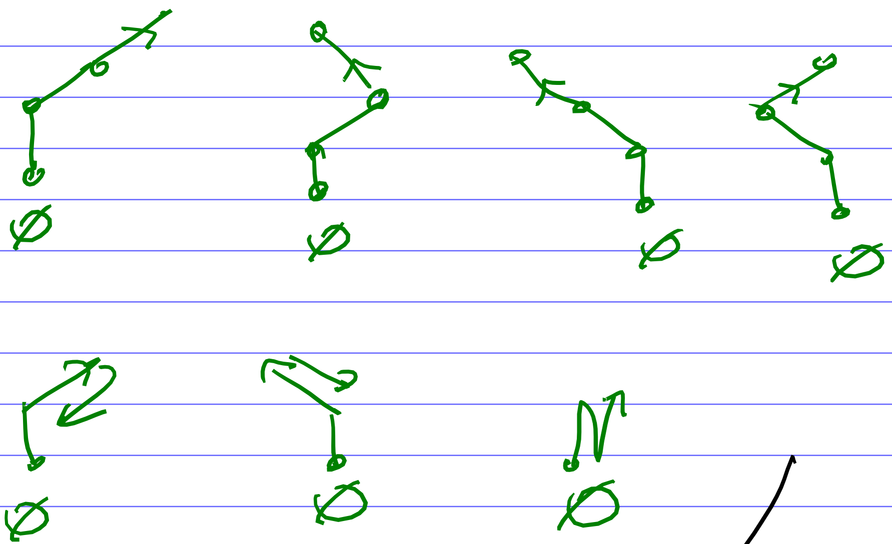
equals

involutions in S_n with all 2-cycles colored in 2 colors.

Example $n = 3$

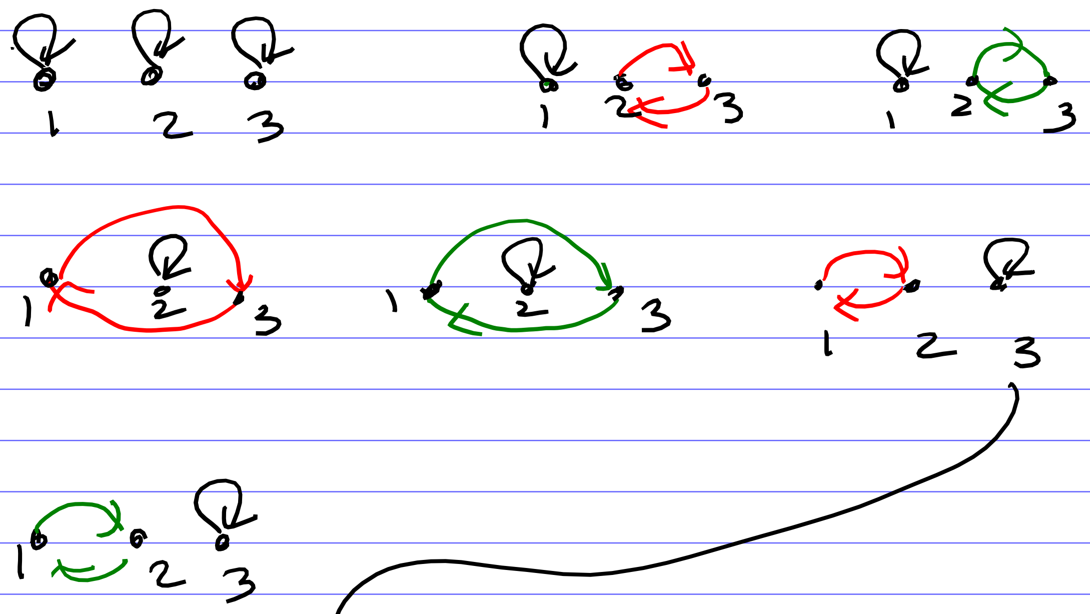
we need to count all paths in this graph starting at $\varnothing$ with 3 steps





7 paths (oscillating tableaux)

Colored involution in S_3



7 colored involutions

Exercise, Find an explicit formula (involving a summation) for this number.