FIRST FUNDAMENTAL THEOREM OF WELFARE ECONOMICS

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ABSTRACT. Markets are a basic tool for the allocation of goods in a society. In many societies, markets are the dominant mode of economic exchange. In this paper, we will prove the first fundamental theorem of welfare economics, which provides a theoretical justification for the efficiency of markets. Roughly speaking, we show that under particular assumptions, markets will tend toward efficient allocations of resources.

1. INTRODUCTION

1.1. Markets and Fair Division.

A *market* is a system in which parties exchange goods and services. Market mechanisms enable the allocation of resources in a society through the establishment of prices for goods and services, which are determined by the forces of supply and demand. Whether markets allocate resources well in a society is an open question of great disagreement among economists.

As such, the study of markets in economics is intricately tied to *fair division problems* which have long been of great interest in mathematics. Such problems ask: how can a set of goods be fairly divided among multiple parties? Of course, fair division problems are of great practical significance, and markets are a real-world mechanism by which goods and services can be allocated among multiple parties.

There exist many types of fair division problems, which depend on diverse factors including the nature of participant preferences, the types of goods being divided, and the desired fairness criteria. Research into fair division problems has encompassed a number of approaches, including studies into the existence or non-existence of fair divisions, of

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the properties of fair divisions, and of algorithms to produce fair divisions.

In such problems, actors assign different values to different goods, according to their own utility functions. The types of goods to be allocated can have a variety of properties. They can be *indivisible* items (e.g. buildings, cars, and paintings) or *divisible* resources (e.g. cake, which can be sliced into pieces of arbitrary size). Goods can be *homogeneous* (e.g. money, for which only the amount matters) or *heterogeneous* (e.g. cake, in which different slices have differing properties). Market mechanisms can produce an allocation for goods of all types.

1.2. **Efficiency in Markets.**

There exist many criteria by which the desirability of a division can be assessed. Accordingly, market success can be measured by the desirability of the produced allocation. As the very name suggests, fair division problems in mathematics have focused mainly on desirable fairness properties of resulting divisions, such as *proportional division* (in which all participants get at least $\frac{1}{N}$ of the total value of the goods, by their own valuation), *envy-free division* (in which no participant prefers another's share over their own), and *equitable division* (in which all participants feel the same happiness, by their own subjective valuation).

In economics, however, the notion of *efficiency* has been the primary criteria by which market success in the allocation of goods is assessed. In particular, *market efficiency* is taken to be the property of *Pareto optimality*: under a *Pareto optimal* divison, it is impossible to make an individual party better off without making another worse. Understood in other words, under divisions that are *not* Pareto optimal, it is possible to re-allocate such that all parties are better off. This is a desirable property of allocations that is not captured by aforementioned fairness criteria: in "fair" divisions, it is often the case that all parties can be made better-off simultaneously.

Roughly speaking, the *first fundamental theorem of welfare economics* states that competitive markets will tend toward equilibria of efficient allocations. It serves as a theoretical justification for the efficacy of markets.

2. Definitions

To formally state the *first fundamental theorem*, we will first need to define a market, as well as formalize notions of *local nonsatiation of preferences*, *Pareto optimality*, and *competitive equilibrium*.

We start by defining a market.

Definition 2.1. A *market* is an environment of production, consumption, and exchange containing the following:

(1) M different goods.
  - Let $p \in (\mathbb{R}^+)^M$ denote the vector of prices for the goods, with p_m denoting the unit price of good m.

(2) N consumers.
For each consumer i ($i \in \{1, 2, ..., N\}$)...
  - let $e_i \in \mathbb{R}^M$ denote his/her starting endowment (the amount of each good owned initially).
  - let $u_i : \mathbb{R}^M \to \mathbb{R}$ denote his/her utility function over his/her consumption of each good.
  - let $x_i \in \mathbb{R}^M$ denote how much of each good is consumed, yielding utility $u_i(x_i)$.

(3) F firms.
  - For each firm $j \in \{1, 2, ..., F\}$, let $Y_j \subset \mathbb{R}^M$ be its *production set*, or the set of all feasible production plans.
  - Let $y_j \in Y_j$ denote the actual production plan chosen by firm j.
  - For example, if $M = 3$ and the production plan $(1, -1, -1) \in Y_j$, firm j has the ability to produce one unit of good 1 from one unit of good 2 and one unit of good 3.
  - The firms are owned by consumers.
    - If consumer i owns a share θ_{ij} of firm j, we have $\sum^N i = 1\theta_{ij} = 1$ for each j.
    - If firm j produces y_j at prices p, it profits $\pi_j = p \cdot y_j$ and pays out $\theta_{ij}\pi_j$ to consumer i.

Next, for the first fundamental theorem to apply, it is required that consumer preferences are *locally nonsatiable*.

Definition 2.2. Consumer preferences are *locally nonsatiable* if for all consumers i and their consumption bundles x_i, for any $\epsilon > 0$, there exists some x_i' such that $\|x_i' - x_i\| \leq \epsilon$ and $u_i(x_i') > u_i(x_i)$.

This condition enforces that there is always some way to marginally improve a consumer's consumption bundle. Intuitively, this is a reasonable assumption, given that consumers are constrained by wealth and there presumably exists a marginally more expensive consumption bundle that is unafforable, yet yields higher utility.

Next, let us rigorize our notion of efficiency by formally defining *Pareto optimal*.

Definition 2.3. Let $x = (x_1, x_2, ..., x_N) \in \mathbb{R}^{M \times N}$ be the consumption plans for each consumer. Call x *feasible* if it is possible to produce enough of each good in the market: i.e., if there exists a production plan $(y_1, y_2, ...y_F) \in Y_1 \times Y_2 \times ... \times Y_F$ such that

$$\sum_{i=1}^{N} x_i \leq \sum_{i=1}^{N} e_i + \sum_{j=1}^{F} y_j$$

Call a feasible plan x *Pareto optimal* if it is not *Pareto dominated* by another feasible consumption plan: that is, there does not exist a feasible plan $x' = (x'_1, x'_2, ..., x'_M)$ such that $u_i(x'_i) \geq u_i(x_i)$, for every i and strict inequality holding for at least one i.

Finally, we shall define the notion of *competitive equilibrium*. Intuitively, a market is at competitive equilibrium if two conditions are met. First, firms and consumers must be "price takers" who must accept market prices and behave optimally conditional on those prices. This means that we have a competitive market without monopolies or monopsonies in which either firms or consumers have the power to affect prices. Second, prices are such that demand equals the supply for each good. We can understand this condition as defining an "equilibrium," and the first condition as defining "competitive."

Formally, we have the following.

Definition 2.4. A *competitive equilibrium* is a price vector p^*, a consumption plan for each consumer $x^* = (x_1^*, ..., x_N^*)$, and a production plan for each firm $y^* = (y_1^*, ..., y_F^*)$ such that:

(1) Given price p^*, each firm is maximizing profits. That is, for each $j \in 1, 2, ..., F$,

$$y_j^* \in \arg\max_{y_j \in Y_j} p^* \cdot y_j.$$

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(2) Given prices p^* and their wealth (comprising both initial endowment and income from firm ownership), each consumer maximizes utility. That is, for each i, we have that
$$x_i^* = \arg\max_{x\ i}\{u_i(x_i) : p^* \cdot x_i \leq p^* \cdot e_i + \sum_j \theta_{ij}(p^* \cdot y_j^*)\}.$$
(3) Supply for each good equals demands for each good. That is,
$$\sum_i x_i^* = \sum_i e_i + \sum_j y_j^*.$$

3. First Fundamental Theorem of Welfare Economics

Now, we are ready to state our main result.

Theorem 3.1. *(The First Fundamental Theorem of Welfare Economics). If (p^*, x^*, y^*) is a competitive equilibrium in a market in which consumers have locally nonsatiable preferences, x^* is Pareto optimal.*

We first show the following result which will be used to prove the first fundamental theorem.

Lemma 3.2. *At any competitive equilibrium (p^*, x^*, y^*), a consumption plan x' that dominates x^* must be more expensive than x^*.*

Proof. Since x' strictly dominates x^*, at least one consumer strictly prefers x_i' to x_i^*. Let this consumer be consumer k. Since consumer k chose x_k^* at prices p^*, and consumers maximize their utility in equilibrium, x_k^* must have been the best consumption plan he/she could afford. Since x_k' is a better consumption plan, it must be unaffordable. Put mathematically,
$$p^* \cdot x_k' > p^* \cdot x_k^*.$$

Since x' Pareto dominates x^*, for all other consumers, it must be the case that x_i' is at least as good as x_i^*. We can show that this implies that x_i' is at least as expensive as x_i^* at prices p^*.

Suppose it were false. Then for some i, we must have that $u_i(x_i') \geq u_i(x_i^*)$ but $p^* \cdot x_i' < p^* \cdot x_i^*$. Here, we apply the property of local nonsatiation of consumer preferences. Let $\delta = p^* \cdot x_i^* - p^* \cdot x_i'$, and let $\epsilon = \frac{\delta}{\|p^*\|}$. By local nonsatiation, there exists an x_i'' such that $\|x_i'' - x_i'\| \leq \epsilon$ and $u_i(x_i'') > u_i(x_i')$. Intuitively, there is an arbitrarily close consumption plan that is marginally better.

So, we have that

$$p^* \cdot x_i'' = p^* \cdot (x_i'' - x_i') + p^* \cdot x_i' = p^* \cdot (x_i'' - x_i') + (p^* \cdot x_i^* - \delta).$$

Applying the basic vector inequality $a \cdot b \leq \|a\|\|b\|$, we have that

$$p^* \cdot (x_i'' - x_i') \leq \|p^*\|\epsilon = \|p^*\| \frac{\delta}{\|p^*\|} = \delta$$

Therefore,

$$p^* \cdot x_i'' \leq \delta + (p^* \cdot x_i^* - \delta) = p^* \cdot x_i^*,$$

implying that x_i'' is an affordable consumption plan. But note that we have $u_i(x_i'') > u_i(x_i') \geq u_i(x^*)$, or in other words, that the utility of consumption plan x_i'' is strictly higher than that of x_i^*, despite being affordable. But by definition, x_i^* was chosen in equilibrium, meaning that x_i'' cannot be affordable and offer strictly higher utility, as consumers are utility-maximizing.

Therefore, we have a contradiction, and it must have been the case that x_i' is at least as expensive as x_i^* at prices p^*, or $p^* \cdot x_i' \geq p^* \cdot x_i^*$. Summing up over all consumers, we have the result that the consumption plan x' must be more expensive than x^*:

$$\sum_{i=1}^{N} (p^* \cdot x_i') > \sum_{i=1}^{N} (p^* \cdot x_i^*).$$

□

Now, we can complete the proof of our main result.

Proof. (Theorem 3.1) We proceed with a proof by contradiction. Assume that there is a competitive equilibrium (p^*, x^*, y^*) which is not Pareto optimal. By Lemma 3.2, at prices p^*, any feasible consumption plan $x' = (x_1', x_2', ..., x_N')$ that dominates x^* must be more expensive than x^*. Mathematically,

$$\sum_{i=1}^{N} (p^* \cdot x_i') > \sum_{i=1}^{N} (p^* \cdot x_i^*). \tag{3.1}$$

Next, let the production plan $y' = (y_1', y_2', ..., y_F')$ generate the feasible consumption plan x'. This gives us that $\sum_i x_i' = \sum_i e_i + \sum_j y_j'$. And further, since supply equals demand in our market at equilibrium, we have that $\sum_i x_i^* = \sum_i e_i + \sum_j y_j^*$.

Plugging into equation (3.1), we have:

$$p^* \cdot (\sum_i e_i + \sum_j y_j') > p^* \cdot (\sum_i e_i + \sum_j y_j^*).$$

Subtracting $p^* \cdot \sum_i e_i$ from both sides and placing the dot product back in the summation, we have:

$$\sum_j p^* \cdot y_j' > \sum_j p^* \cdot y_j^*.$$

But since the left hand side of the equation is strictly greater than the right, at least one of its summands must be larger than the corresponding summand in the right hand side. In other words, there must exist a firm k such that:

$$p^* \cdot y_k' > p^* \cdot y_k^*.$$

However, this is a contradiction. By the definition of competitive equilibrium, firm k chose the production plan y_k^* at p^* to maximize profits. However, y_k' is a feasible production plan that generates higher profits. So, competitive equilibria must be Pareto optimal.

□

4. Interpreting the Theorem

We have now proven the main result of this paper, the first fundamental theorem of welfare economics. Next, we give some additional background with respect to the interpretation of the theorem.

4.1. **Existence of Equilibria.**

By the first fundamental theorem of welfare economics, any competitive equilibrium in a market must be Pareto optimal. However, this does not guarantee the existence of such an equilibrium in all markets. It is possible to prove the existence of a competitive equilibrium with some additional assumptions, although this proof is beyond the scope of this paper. For example, in a pure exchange economy with only consumers and no firms, the existence of a competitive equilibrium is guaranteed if the following conditions are met: (1) each utility function u_i is continuous, (2) each u_i is increasing, (3) each u_i is concave, and (4) every consumer has a strictly positive endowment of every good. If any of these conditions is not met, it is no longer guaranteed that there exists a equilibrium.

We can give a simple example of a market in which there exists no competitive equilibrium.

Example 4.1. Consider a market with two goods, two consumers, and no firms. Let $u_1(x, y) = \min\{x, y\}$ and $u_2(x, y) = \max\{x, y\}$ (which is a convex function). Let the initial endowments for the two consumers be $e_1 = e_2 = (1, 1)$.

Note that at positive prices, consumer 1 will refuse to trade since he/she would like to own an equal quantity of the two goods. However, consumer 2 would like to trade all of one good for as much of the other good as possible. This means that at positive prices, supply cannot possibly equal demand. If the price of either good is 0, then consumer 2 demands an infinite quantity of that good, meaning that it is also impossible for supply to equal demand. In both the case of positive and non-positive prices, it is impossible for supply to equal demand. Therefore, an equilibrium cannot exist.

4.2. **Second Fundamental Theorem of Welfare Economics.**

The first fundamental theorem of welfare economics guarantees that any competitive equilibrium is Pareto optimal. However, there may exist multiple competitive equilibria, with some more desirable than others. For example, one possible Pareto optimal competitive equilibrium in a pure exchange market is a final allocation such that a single consumer owns all the goods. Assuming that consumer's utility function to be increasing, under such a scenario, it is impossible to improve the utility of any other consumer without making that consumer worse off.

This equilibrium allocation, although Pareto optimal, is intuitively undesirable. It certainly does not meet any reasonable fairness criteria. However, the second fundamental theorem of welfare economics holds under some further conditions. It gives us some additional hope for markets.

Theorem 4.2. *(The Second Fundamental Theorem of Welfare Economics). In a market, any Pareto optimal consumption plan is a competitive equilibrium for some set of initial endowments.*

The proof of the second fundamental theorem is beyond the scope of this paper. However, for the theorem to hold, we need additional assumptions of continuous, concave utility functions, as well as convex productions sets for firms (although these are reasonable assumptions). The theorem intuitively tells us the following: any Pareto optimal distribution can be achieved by implementing redistributive transfers on the initial endowments, and then allowing the market to take over.

Consider that we should intuitively prefer that allocations of goods and resources in society be Pareto optimal. Otherwise, it would simply be possible to make all consumers in the society simultaneously better-off. The second fundamental theorem tells us that through appropriate redistribution of initial endowments, we can pick the Pareto optimal competitive equilibrium at which the market arrives. Taken together with the first fundamental theorem, we have a much more powerful justification for the efficacy of markets than the first fundamental theorem can offer alone.

4.3. **Applicability of the Theorems.**

The applicability of the fundamental theorems is an open question among economists. The first fundamental theorem has been interpreted some by economists as a theoretical justification of Adam Smith's "invisible hand" hypothesis: the self-interested actions of actors in a market lead to unintended social benefits, and ultimately, desirable economic outcomes.

Other questions have put question to the realisticness of the assumptions underlying the fundamental theorems. For example, human beings are not perfectly rational actors, and it is only approximately true that consumers are utility-maximizing and firms are profit-maximizing. Further, in real markets, consumers and firms are not always simply "price takers." There often exist monopolies (single firm) or monopsonies (single consumer) in which either firms or consumers have great power in affecting, or even setting prices. The degree to which markets diverge from the assumptions behind the fundamental theorems determines the extent to which the fundamental theorems are applicable.

Finally, one powerful way in which the fundamental theorems fail is that they do not account for externalities, or costs or benefits to third-parties who did not choose to incur such costs or benefits. For example, a consumer who chooses to consume one gallon of gasoline will pollute the environment and lower the utility of every other consumer. The fundamental theorems, however, assume that a consumer's utility is solely a function of that consumer's consumption bundle. It does not account for the possibility that one's utility can be affected by the consumption bundle of another consumer.

Ultimately, despite the positive case for markets that can be made with the fundamental theorems, and the possible ways in which real markets can diverge from idealized assumptions, it is the case that market economies in the world are empirically more prosperous than non-market economies.

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