THE FRIENDSHIP THEOREM

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Abstract. In this paper we explore the friendship theorem, which in graph theory is stated as if we have a finite graph in which any two nodes have exactly one common neighbor, then Then there is a node which is adjacent to all other nodes. We provide a common proof of the friendship theorem, followed by two extensions. The first extension relates to the number of common neighbors a node must have. The second relaxes the friendship condition such that any two nodes can have no common neighbor or one common neighbor.

1. Introduction

The Friendship Theorem has long been explored through graph theory, and its origins are not known. The Friendship Theorem can be stated as follows:

Theorem 1.1. *Suppose in a group of people we have the situation that any pair of persons has precisely one common friend. Then there is always a person (the "politician") who is everybody's friend.*

In such a situation, there is always one person who is everyone's friend. We will see throughout this paper that the Friendship Theorem can be explored with graph theory in order to provide valuable insights to a variety of graphs of different diameter. We will summarize a simple proof of the friendship theorem as well as a purely combinatorial proof proposed by Mertzios and Unger in 2008, and a relaxation of the problem and further conclusions discussed by Skala in 1972.

The friendship theorem is commonly translated into a theorem in graph theory:

Theorem 1.2. *Suppose that G is a finite graph in which any two vertices have precisely one common neighbor. Then there is a vertex which is adjacent to all other vertices. [1,2]*

An example of a graph adhering to the properties defined in this problem can be seen in figure 1. The friendship condition can be restated as "For any pair of nodes, there is exactly one path of length

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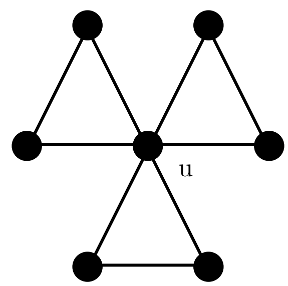

FIGURE 1. A windmill graph with politician u

two between them". [4]

In addition to these common expressions of the Friendship Theorem, we will discuss an extension of the Friendship Theorem which works off of the set theory translation of the theorem:

Theorem 1.3. *A friendship set is a finite set with a symmetric non-reflexive binary relation, called on, satisfying: (i) if a is not on b, then there exists a unique element which is on both a and b; (ii) if a is on b, then there exists at most one element which is on both a and b. [5]*

2. A Proof of the Friendship Theorem

In this section, we will show a traditional proof of the Friendship Theorem thought to be first published by Paul Erdos, Alfred Renyi, and Vera Sos. We begin with a discussion of some simple properties which we will refer to throughout the paper, then move into the proof.

Proposition 2.1. *A friendship graph G adheres to the C_4-condition, that is, it contains no cycle on four nodes. Moreover, the distance between any two nodes in G is at most two.*

Proof. Suppose G includes C_4 as a subgraph. There are two nodes v and u which have at least two common neighbors, as two opposite vertices would in a square. This is in contradiction to the friendship condition. The distance between two nodes must be at most two, because if it is greater than two then they do not have any "friends" in common, also violating the friendship condition. □

We will prove Theorem 1.2 (and thus the other translations of the Friendship Theorem) by contradiction, supposing it is false for some graph G. This means that there is no vertex of G that is adjacent to all other vertices.

Proof. Suppose we have a graph G that fits the friendship condition. G must be a regular graph, so that the degree of each vertex is equal. We will prove this combinatorially. Any two nonadjacent vertices u and v must have equal degree $d(u) = d(v)$.

Suppose $d(u) = k$ and $w_1, ...w_k$ are adjacent to u. One of these adjacent vertices, say w_2 must also be adjacent to v in order to satisfy the conditions of the Friendship Theorem. In addition w_2 must be adjacent to one other vertex in $w_1, ...w_k$, say w_1. Thus, v must be adjacent to some other vertex in common with w_2 in order to satisfy the theorem and the $C_4 - condition$. For all w_i adjacent to u except w_1, v must have a common neighbor and this neighbor must be distinct in order to avoid a cycle of length 4. Thus, $d(v) \geq k = d(u)$, and by symmetry $d(u) = d(v) = k$.

We have constructed the graph such that w_1 is the only shared neighbor of u and v, so any other vertex is adjacent to at most one of u and v, and the degrees of all verticies are equal, and equal to k.

The total number of vertices can be counted now that we know the degree of each vertex is k. We count the degree of each w_i adjacent to u to obtain k^2, then subtract $k - 1$ as we have counted u a total of k times. This gives us the number of vertices $n = k^2 - k + 1$.

We continue the proof with applications of linear algebra to arrive at a contradiction. At this point we can conclude that k must be greater than 2 as $k \leq 2$ yields trivial windmill graphs, and we are looking to prove that there are graphs apart from windmill graphs that satisfy the friendship condition.

Consider an adjacency matrix A. This matrix will have k one's in any row as the degree of each vertex is k. By the friendship condition, any two rows have one column where they both have a one. Therefore, A^2 will have k's down the diagonal and 1's everywhere else. To find the eigenvalues of A^2 we write $A^2 = J + (k - 1)I$. From this equation we see that the eigenvalues of A^2 are $n + k - 1 = k^2$ and $k - 1$. Therefore, the eigenvalues of A of k and $\pm\sqrt{k-1}$. We know the multiplicity of k is 1, and suppose that the other eigenvalues have multiplicity r and s such that $r + s = n - 1$.

From the eigenvalues and the trace of A, which is equal to 0, we obtain the equation

$$k + r\sqrt{k-1} - s\sqrt{k-1} = 0$$

which allows us to see that $r \neq s$ as $k \neq 0$. Factoring and squaring this equation yields

$$(r - s)^2(k - 1) = k^2$$

Thus $(k - 1)$ divides k^2, which is only true for $k = 2$. Here we find our contradiction, as the first part of the proof concluded that $k > 2$.

Thus, our graph must be a windmill graph with one vertex of degree $n-1$ in order to satisfy the friendship condition. □

3. The Generalized Friendship Problem

As stated previously, we can rewrite the friendship problem in order to address the diameter of a graph as "For every pair of nodes, there is exactly one path of length two between them." We can generalize this problem to be variable on number of paths between nodes and the length of said paths to achieve some interesting results. We call graphs with path length k and number of paths between nodes l l-regularly k-path connected graphs, or simply $P_l(k)$-graphs. The friendship theorem we have proved in section 2 relates to $P_1(2)$-graphs. [4]

3.1. **Kotzig's Conjecture.**

One may wonder what happens when we alter k, the length of paths in a graph, while keeping $l = 1$, so that between any pair of nodes, there is one path of length k between them. In 1974, Kotzig conjectured that there exists no $P_1(k)$-graphs where $k > 2$. He was able to prove this for $k \leq 8$ by using statements on even cycles, and more cases have been proven up through $k \leq 33$, however a general proof for all cases of k has not yet been formulated.[1]

3.2. **L-Friendship Graphs.**

We now generalize on l, keeping a path length of 2, calling these graphs l-friendship graphs or $P_l(2)$-graphs. These graphs satisfy the condition that every pair of nodes has exactly l common neighbors. [4]

Lemma 3.1. *Every l-friendship graph G is a regular graph for $l \geq 2$*

Proof. We have graph G with set of vertices V. Consider a node v with degree d, and denote L as the subgraph of G consisting of the neighbors of v, and L' as the subgraph of G consisting of all $V\ L, v$. It follows that every node in L' has distance 2 from v. Consider a second node $a \in L$. This node has l neighbors in common with v, and thus l neighbors in L.
Case 1: $L' = \emptyset$. In this case, a has l neighbors in L, plus v as a neighbor, and therefore degree $l+1$. This is true for all of a's neighbors, $a_1, a_2, ...a_l$ as well. We know that v has degree of at least $l+1$ to account for a and a's l neighbors. If $d > l+1$, L must contain some other node b. b would have no neighbors in common with any $a, a_1, a_2, ...a_l$ as this would violate the condition that each a_i has l neighbors in l. Thus the pair a, b would have only one node in common, v, which violates the l-friendship condition. Therefore, b cannot exists and all nodes in L have degree $l+1$. As $V = L \cup v$, all nodes in V have degree $l+1$ and

G is a regular graph.
Case 2: $L' \neq \emptyset$. Now, every node $x \in L'$ has l neighbors in L so that v and x have l common neighbors. For some $a \in L$, there are $(d-1)$ nodes $b \in L$ that are not a, and pairs a, b have l common neighbors, so there must be $(d-1)l$ paths from a to any extreme node $b \in L$. Of these paths, $d-1$ go through v, and $l(l-1)$ go through nodes in L. We derive from this that $(d-l-1)(l-1)$ of the paths go through some node in L'.

We now consider $c \in L'$ that is a neighbor of a, which must be an intermediate node in $l-1$ paths of length 2 from a to some node $b \in L$ as x has $l-1$ neighbors in L that are not a. It follows that a has

$$\frac{(d-l-1)(l-1)}{(l-1)} = (d-l-1)$$

neighbors in L'. We can now conclude that any node in L has

$$1 + l + (d-l-1) = d$$

neighbors. Next we must prove the same for each vertex in L', and we do this by showing that we can select any node as v to create any subsets L and L' which have the same properties as those we have discussed. We obtain the equation

$$|L'| = \frac{d(d-l-1)}{l}$$

from the fact $|L| = d$ and each node in L has $(d-l-1)$ neighbors in L' and each node in L' has l neighbors in L. It follows that

$$|V| = |L| + |L'| + 1 = \frac{d(d-1)}{l} + 1$$

This will hold for any $v \in V$ of degree d, so G must be a d-regular graph. □

4. A Relaxation of Friendship

In this section, we will discuss a relaxation of the conditions of the traditional problem suggested by H.K. Skala. Recall Theorem 1.3, which offers a translation of the problem into sets with the "on" relation, which we will refer to as θ. For clarity, a on b will be written as $a\theta b$ and a not on b will be written as $a\theta' b$. We will relax the friendship condition discussed throughout this paper so that two friends do not need a common friend. In the set representation of a relaxed friendship, friendship set A satisfies the following conditions:

(1) $a\theta' a, \forall a \in A$
(2) if $a\theta' b, \exists c \in A$ such that $a\theta c$ and $b\theta c$

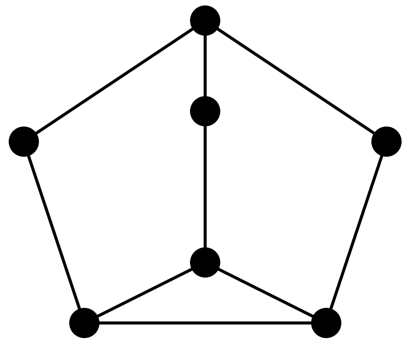

FIGURE 2. A graph representation of an nontrivial friendship set

(3) if $a\theta b$, there exists at most one c such that $a\theta c$ and $b\theta c$

A friendship set A is nontrivial if there is no politician, or if it does not satisfy the conditions presented by the strict friendship condition referenced in Section 1. Skala also discusses in his extension the concept of a P-element. In particular, an element $a \in A$ is a P-element in $a\theta x$ for all elements $x \in A$, $x \neq a$. A nontrivial friendship set would have no P-elements. Figure 2 an example of a nontrivial friendship set. Similarly, an element in a friendship set is called a k-element if it is on exactly k other elements.

Lemma 4.1. *Every element of a nontrivial friendship set is on at least two distinct elements.*

Proof. The fact that each element is on at least one element is trivial by the definition of the friendship set. Assume for the sake of contradiction that some element a in a nontrivial friendship set A is on only one element b. There must exist other elements in the set, and they must have an element in common with a by (2). However, this would make b a P-element, contradicting the notion that A is nontrivial. □

Lemma 4.2. *If a and b are in a nontrivial friendship set A and $a\theta b$, then there exists x in A such that $a\theta' x$ and $b\theta' x$.*

Proof. Suppose for the sake of contradiction that there exists x such that x is on either a or b. To avoid an element being a P-element, there must be at least two other elements in A, call them c and d, such that $c\theta' a$ and $d\theta' b$. We know that $c\theta' d$ by the C_4-condition. Elements c and d must have a different common element e, either on a or b, but e cannot be on a or b by the C_4-condition, so we have reached a contradiction. □

Skala's paper continues with fairly complex proofs of the following lemma and theorem through both combinatorics and linear algebra.

Lemma 4.3. *Let a and b be k_1- and k_2-elements of a nontrivial friendship set A and suppose $k_1 \geq k_2$. If $a\theta' b$ then $k_1 = k_2 - 1$ or $k_1 = k_2$; moreover, if $k_1 = k_2 - 1$, then there is an element on bot a and the common element between a and b. If $a\theta b$ and there exists an element on both a and b, then $k_1 = k_2$. [5]*

Theorem 4.4. *If A is a finite nontrivial friendship set, then there exists an integer m such that every element is an m- or an $(m-1)$-element. [5]*

Theorem 4.5. *If m is the degree of a homogeneous friendship set A, then A contains $m^2 + 1$ elements; m can be only 2, 3, 7 or 57. [5]*

This is an interesting result that, when thought about in terms of the graphical representation of Skala's friendship sets, shows a fundamental difference between his generalization of the friendship condition and the generalization in section 3. In particular, for l-friendship graphs must be regular graphs, whereas Skala's sets can only be represented by regular graphs if the degree of each node is 2, 3, 7, or 57.

5. Conclusions

In this paper we provide a proof of the friendship theorem as well as two extensions on the theorem and a discussion of Kotzig's conjecture in relation to the friendship theorem. There are certainly other problems left open, such as the generalization of $P_l(k)$-graphs as a class with variation on both l and k.

References

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