Uncovering the Lagrangian of a system from discrete observations

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The Problem

Given discrete measurements of a Lagrangian system, can we recover the Lagrangian?

The Discrete Euler-Lagrange Equations

Given a system with Lagrangian $L(x,v)$, we can discretize the action with time step $\tau$ by summing the discrete Lagrangian $L_d(x,y) = \tau \cdot L(x + y^2, y - x\tau)$.

The principle of stationary action yields the discrete Euler-Lagrange equations $D^2 L_d(x,y) + D_1 L_d(y,z) = 0$, relating any three consecutive points $x$, $y$, and $z$ on a discrete trajectory.
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Given a system with Lagrangian $L(x, \nu)$, we can discretize the action with time step $\tau$ by summing the discrete Lagrangian

$$L_d(x, y) = \tau \cdot L \left( \frac{x + y}{2}, \frac{y - x}{\tau} \right).$$
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Given a system with Lagrangian \( L(x, \nu) \), we can discretize the action with time step \( \tau \) by summing the discrete Lagrangian

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relating any three consecutive points \( x, y, \) and \( z \) on a discrete trajectory.
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Given a pair of points \((x_0, y_0)\), we would like to use data points on trajectories that pass nearby to estimate the Taylor expansion of the discrete Lagrangian \(L_d\) at \((x_0, y_0)\).
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A Caveat

Two Lagrangians might be equivalent in the sense that they yield the same equations of motion.
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Two Lagrangians might be equivalent in the sense that they yield the same equations of motion.

- For example, if \(L(x, y)\) is a discrete Lagrangian, then the Lagrangian

\[
L'(x, y) = \alpha L(x, y) + \beta (y^2 - x^2) + \gamma (y - x) + \delta
\]

produces the same discrete Euler-Lagrange equations, for any choice of \(\alpha, \beta, \gamma, \text{ and } \delta\).
Recovering the Discrete Lagrangian

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- Trajectory data can’t distinguish between equivalent Lagrangians, nor would it be useful to do so.
A Second-Order Approximation

Taylor Expansion of the Discrete Lagrangian

Given a pair of points \((x_0, y_0)\), we approximate \(L_d\) with its second-degree Taylor polynomial at \((x_0, y_0)\).

\[
L_d \approx a(x - p)^2 + 2b(x - p)(y - p) + c(y - p)^2 + dp(x - p) + ep(y - p) + f p,
\]

where \(p = (x_0 + y_0) / 2\).

Discrete Euler-Lagrange Equations

For three consecutive points \(x, y,\) and \(z\) on a trajectory, we can apply the discrete Euler-Lagrange equations

\[
D^2 L_d(x, y) + D_1 L_d(y, z) = 0
\]

\[
0 \approx 2(a + c)(y - p) + 2b(x - p + z - p) + (dp + ep).
\]

We will use nearby triplets \((x, y, z)\) from our trajectory measurements to estimate \(a + c, b,\) and \(d + e\).
Taylor Expansion of the Discrete Lagrangian

Given a pair of points \((x_0, y_0)\), we approximate \(L_d\) with its second-degree Taylor polynomial at \((x_0, y_0)\). We rewrite it in the form

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We will use nearby triplets \((x, y, z)\) from our trajectory measurements to estimate \(a + c, b,\) and \(dp + ep\).
In order to estimate the parameters using many data points, we need to assign weights to the parameters appropriately.

Taylor approximations of the discrete Euler-Lagrange equations suggest that an appropriate rescaling of the parameters at \((x_0, y_0)\) is

\[
A := (a + c) \| y_0 - x_0 \|, \quad B := 2b \| y_0 - x_0 \|, \quad D := dp + ep.
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Scaling the Parameters

- In order to estimate the parameters using many data points, we need to assign weights to the parameters appropriately.
- The parameter $d_p + e_p$ has different units from $a + c$ and $b$. 

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Discrete Euler-Lagrange Equations

For three consecutive points $(x, y, z)$ on a trajectory, we have

$$0 \approx A(2y - x_0 - y_0) + B(x + z - x_0 - y_0) + D \|y_0 - x_0\|.$$
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New Parameters for the Lagrangian

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Estimating Lagrangian Parameters with Several Data Points

Given data of consecutive triplets \((x_i, y_i, z_i)\), we estimate \(A, B,\) and \(D\) up to scaling at a point \((x_0, y_0)\) as follows.
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- Construct a matrix \(M\) whose rows are

\[w_i \cdot (2y_i - (x_0 + y_0)) \quad x_i + z_i - (x_0 + y_0) \quad \|y_0 - x_0\|\].
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- For the correct values of the parameters, we will have \(M \begin{pmatrix} A \\ B \\ D \end{pmatrix} \approx 0\).
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- For the correct values of the parameters, we will have 
  \[M \begin{pmatrix} A \\ B \\ D \end{pmatrix} \approx 0.\]

- Estimate \(A, B,\) and \(D\) by the eigenvector corresponding to the least eigenvalue of \(M^T M\).
Assigning Weights to the Data Points

The matrix of coefficients

The $i$th row of $M$ is

$$w_i \cdot (2y_i - (x_0 + y_0) \cdot x_i + z_i - (x_0 + y_0) \parallel y_0 - x_0\parallel).$$
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Distance

We define the distance between $(x_0, y_0)$ and $(x, y)$ to be

$$\delta((x_0, y_0), (x, y))^2 = \left\| \frac{x+y}{2} - \frac{x_0+y_0}{2} \right\|^2 + \tau_s^2 \left\| \frac{y-x}{\tau} - \frac{y_0-x_0}{\tau} \right\|^2,$$

where $\tau_s$ is a parameter and $\tau$ is the timestep.
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Weights

$$w_i = \exp \left( -\frac{1}{2\sigma^2} \left( \delta((x_0, y_0), (x_i, y_i))^2 + \delta((x_0, y_0), (y_i, z_i))^2 \right) \right) ,$$

where $\sigma$ is another parameter.
The Simple Pendulum

The Lagrangian

\[ L_d(x, y) = \tau \left( \frac{1}{2} \left( \frac{y - x}{\tau} \right)^2 - \left( 1 - \cos \left( \frac{x + y}{2} \right) \right) \right). \]
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True Values of Lagrangian Parameters

Using a Taylor approximation to the Lagrangian, we find that

\[
\frac{B}{A} = -\frac{4 + \tau^2 \cos \left( \frac{x_0 + y_0}{2} \right)}{4 - \tau^2 \cos \left( \frac{x_0 + y_0}{2} \right)}, \quad \frac{D}{A} = -\frac{4\tau^2}{\|y_0 - x_0\|} \cdot \frac{\sin \left( \frac{x_0 + y_0}{2} \right)}{4 - \tau^2 \cos \left( \frac{x_0 + y_0}{2} \right)}. \]
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\end{align*}
\]

Parameters Computed From Trajectories

I computed the parameters from the trajectories with Matlab. The graphs of \( \frac{B}{A} + 1 \) and \( \frac{D}{A} \|y_0 - x_0\| \) are on the following slides.
Pendulum simulation at high energy

\[ B/A+1 \]

\[ D||y - x_0||/A \]

Pendulum Position vs. Time

Time

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Pendulum simulation at high energy with noise

\[
B/A + 1
\]

\[
D|y_0 - x_0| / A
\]

Pendulum Position

\[
0 1 2 3 4 5 6 7 8 9 10
\]

\[
-5
0
5
x 10^{-5}
\]

\[
0 1 2 3 4 5 6 7 8 9 10
\]

\[
-1
0
1
x 10^{-4}
\]

\[
0 1 2 3 4 5 6 7 8 9 10
\]

\[
0
20
40
\]

Time

\[
0 1 2 3 4 5 6 7 8 9 10
\]

\[
0
\]

\[
0
20
40
\]

Pendulum Position

\[
0 1 2 3 4 5 6 7 8 9 10
\]

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Pendulum simulation at high energy with noise

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Uncovering the Lagrangian

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What Next?

Future Directions

Recover the Lagrangian from data of several trajectories, and then use it to predict new trajectories.

Investigate the best choices for $\tau$ and $\sigma$.

Try adding other kinds of noise to the system.

Try the method with real data.

Evan S. Gawlik, Patrick Mullen, Dmitry Pavlov, Jerrold E. Marsden, and Mathieu Desbrun, Geometric, variational discretization of continuum theories, 2010.

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