• Euclid’s algorithm, i.e. finding solutions to $ax + by = 1$.
• Legendre transform, in particular finds a map $T^*M \to TM$.
• Tubular neighborhood theorem (neighborhood diffeomorphic to normal bundle)
• $PSL_2(\mathbb{Z}) = \mathbb{Z}_2 \rtimes \mathbb{Z}_3$.
• Schwarz lemma, any biholomorphism of the open disc fixing the origin are rotations.
• The real determinant of a complex matrix is the norm squared of its complex determinant.
• Maximum principle for subharmonic functions. Pseudoholomorphic functions are subharmonic. Much better: Hopf maximum principle
• The investigation of quadratic number fields and their ring of integers is illuminating and beautiful.
• Poincare lemma (holds holomorphically as well).
• Locally free (even free) sheaves are not projective in general.
• Milnor’s curve selection lemma: Let $V \subseteq \mathbb{R}^n$ be an algebraic set, and $U$ be an open set defined by algebraic inequalities. Then for any point in the closure of $U \cap V$ there is an algebraic curve staying inside $U \cap V$, and approaching that point.
• Hodge theory of elliptic complexes - represent baby babey.
• Cech-deRham double complex.
• The main technical point in the proof of fundamental theorem of calculus is the mean value theorem.
• Lie derivative of the vector fields is the second derivative of the commutator of their flows.
• In the locally simply connected case it is very easy to construct the universal cover.
• Eckmann-Hilton argument: if you have two unital products on a set which satisfy $(a \cdot b) \times (c \cdot d) = (a \times c) \cdot (b \times d)$, then they must be the same, and this product must be commutative and associative. Prove that the fundamental group of a Lie group or higher homotopy groups in general are commutative using this.
• Learn to consider vector fields as differential operators. When put together with the concept of $\phi$-relatedness this can be very useful.
• Arzela-Ascoli theorem, Banach-Alaoglu theorem.
• Jordan normal form.
• Sobolev embeddings, also pay attention to the case when the embedding is compact.
• Slices for taking quotients of group actions.
• Lie algebra-Lie group dictionary.
• Picard group of algebraic variety.
• Weierstrass preparation theorem. Take a holomorphic function $f$ in several variables $z, z_1, \ldots, z_n$. Let $f(0, \ldots, 0) = 0$, and also assume that $f$ has some term that only depends on $z$, then we can find an analytic function $h$ which is not zero at the origin, and $W$ which is a monic polynomial in $z$ with coefficients analytic functions in $z_1, \ldots, z_n$, which all vanish at the origin such that $f = h(z)W(z, z_1, \ldots, z_n)$. Hence we can find zeros of $f$ by fixing values of $z_1, \ldots, z_n$ and solving for $z$. So we get $n$ branches and in particular we can’t have isolated zeros.
• Zariski’s lemma. Every finitely generated $k$-algebra which is a field is a finite extension.
• Noether normalization theorem. Best proved geometrically. Take the projectivization of the corresponding variety. One can show that if the affine variety is not the whole of affine space, then there is a point at infinity which is not contained in the projectivization. We choose a hyperplane not containing that point, and project there from that point. The image should be a closed subset of the hyperplane, because projective morphisms are proper. This map should have finite fibers because, the line $\mathbb{C}P^1$ between any point on the hyperplane and the projection point intersects with the variety at a closed subset of the line, but it is not the whole line, so it should be finite. Now continue until the goal is achieved, namely if we go down until the dimension of the variety is the same as the hyperplane, than the image will have to be everything.
• Baire category theorem: Open mapping theorem, closed graph theorem, bounded inverse theorem, Riesz representation theorem.
• The argument principle: Rouche’s theorem, Open mapping theorem for holomorphic maps, maximum principle.
• Perron-Frobenius theorem. If we take a matrix with positive real entries, then it has a positive real eigenvalue, which is strictly maximal in absolute value among all the eigenvalues (possibly complex), and it has a one dimensional eigenspace, which is spanned by an eigenvector with positive real entries. Moreover, there is no other such eigenvector. This has a generalization to bounded operators between Banach spaces, called Krein-Rutmann theorem.
• Riemann-Roch, Grothendieck-Hirzebruch-Riemann-Roch
• Theorem of Grothendieck that every holomorphic vector bundle over the projective line is holomorphically a direct sum of line bundles.
• L’Hospital’s rule
• Kodaira vanishing theorem: Over a compact Kahler manifold of dimension $n$, the $k$th sheaf cohomology of the sheaf of holomorphic $r$-forms with values in an ample line bundle vanishes whenever $k + r > n$. Note particularly the case of $r = n$.
• Period lattice and the Jacobian
• Stokes’ theorem, divergence theorem, Green’s first identity with the Laplacian can be derived from the divergence theorem by writing $\Delta = \text{div}\,\text{grad}$.
• Hopf algebra is (at least) a unification of functions on an affine group scheme (commutative HA) and universal enveloping algebra of a Lie Algebra
• One can compute (at least find information about) periods by analyzing the motive associated to it - essentially by understanding its MHS (Griffiths transversality and more).
• Spherical harmonics give representations of SO(n). Clebsch-Gordon coefficients are simply coefficients of the irreducible representations in tensor products of irreducible representations.
• Alexander-Spanier cohomology
• Dold-Thom theorem (for example: symmetric powers of Riemann surfaces start submersing onto their Jacobian after some point, hence we get a projective space bundle over a torus.)
• Thom’s first transversality lemma (generalization of Ehresmann’s theorem)
• KLN Theorem for infrared divergences.
• It is easy to prove Hodge conjecture in codimension 1 - use the exponential sequence.
• Fundamental theorem of tropical geometry and Viro patchworking.
• One can produce Hodge filtration (not the decomposition) by reduction to finite characteristic - using Hodge to deRham spectral sequence.
• Chen’s theorem computes pro-unipotent completion of the fundamental group of a manifold using iterated integrals of certain nice 1-forms (zeroth cohomology of the bar complex).
• Pham’s generalized vanishing cycles.
• Tropical geometry can compute Hodge numbers.
• The (irreducible) relations between MZV’s are all homogenous, and no one knows why.
• Resolution of diagonal, and Beilinson’s exceptional collection.
• There are blow-up formulas for both the corresponding derived categories and the (Voevodsky) motives.
• Belyi’s theorem.
• Neutral Tannakian category = Representation category of a (non-unique) pro-algebraic group.
• Kontsevich-Zagier conjecture on the equality of periods.
• On a compact Kahler manifold, a holomorphic form is necessarily closed (follows from deedeebar lemma).
• Logarithmic deRham complex can be used to put Hodge structure on a quasi-projective variety which can be compactified using normal crossing divisors at infinity.
• Stability conditions on triangulated categories is a generalization of stable orbits in GIT quotients which corresponds (somehow) to choosing the point in the image of moment map in symplectic reduction. The intermediate step between the two notions is the moduli space of stable G-bundles over say a Riemann surface. There is chamber structure in the moduli space of stability conditions (for example in the the moment polytope) - wall crossing formulas etc. I have to read Mumford - GIT.
• Orderability in contact/symplectic geometry. This is very simple for one dimensional contact manifolds (expectedly of course), and that’s why the overtwisted contact structures in three dimensions were understood much earlier than the higher dimensions.
The plan in constructing contact/symplectic structures from subdivision methods is to first subdivide enough so that the characteristic foliations on the boundaries of top dimensional cells admit a transversal (i.e. an open book decomposition), then the monodromy of the open book is a contactomorphism (in fact a Hamiltonian contactomorphism). Secondly one uses a kind of connect summing procedure to reduce the problem to filling annuli (boundaries with the transverse foliations) rather than filling spheres. This filling question can then be turned into a question about the orderings of the contactomorphisms at the boundary using the trick of realizing the characteristic foliations inside the manifold cross $T^*S^1$ by taking the graph of Hamiltonians. Overtwistedness then corresponds to a kind of viral characteristic foliation which can infect every (horizontal) characteristic foliation, i.e. make it less than any other by connect summing.

- Leray spectral sequence exists for all continuous maps, not just Serre fibrations.
- Spectral sequence is really just a tool for computing the homology of a given chain complex with the help of a filtration by chain complexes.
- Cartan-Eilenberg resolution - I like Etingof’s method of constructing this.
- Tensor product in Fukaya category is something like fibrewise adding Lagrangians in an integrable system. Correspondingly zero section is something like $O$. This is sketchy.
- Multiple polylogarithms have very rich algebraic structure. Physicists have very efficient (usually not proven) methods to obtain functional relations among MPL’s.
- A very important step in constructing the abelian category of motives would be to show that the homology of some very explicit chain complex is zero - something like algebraic differential forms with infinite $(-n)$ degrees on $D^\infty$. See Joseph Ayoub’s work.
- Koszul resolution is the king (generalizes the resolution we used for Bezout theorem, ideal sheaf of a divisor etc.).
- Adjunction formula (this is essentially not hard).
- Spin bundle on the quadric in $\mathbb{P}^3$.
- Hodge to deRham spectral sequence (Frolicher) degenerates at $E^1$ page, which induces a Hodge filtration on the deRham cohomology (graded pieces are what appears in the Hodge decomposition). To prove Hodge decomposition one actually needs analysis, but the filtration can be deduced from completely algebraic methods. This is one reason the filtration is more fundamental in algebraic geometry. Another reason could be that abstract definitions of the category of Hodge structures and Mixed HS give nice abelian categories. Yet another one, maybe most important, is Griffiths transversality.
- There is a SSS for all generalized cohomology theories (Atiyah-Hirzebruch spectral sequence).
- Local to global spectral sequence of Ext’s.
- Torelli theorem that an algebraic curve is determined by its Jacobian as a principally polarized abelian variety. This has generalizations to CY’s, which play a very important role in mirror symmetry.
- Bogomolov-Tian-Todorov theorem.
• There are many birational invariants, like $h^{q,0}$, Kodaira dimension.
• It is not known if the Hodge numbers of Calabi-Yau’s are bounded.
• Atiyah class on the existence of analytic (or algebraic) connections. This actually comes up naturally when you try to glue locally defined connections as a Čech cohomology class. Another way to get to it is to consider the extension of vector fields over the endomorphisms of the bundle by the first order differential operators with scalar symbol. A section of this SES is equivalent to having a connection.
• Zig-zags in the category of dg-algebras is exactly the same as $A_\infty$-maps.
• Kempf-Ness theorem
• Moduli space of polygons as a symplectic reduction.
• If $X$ is simply connected, $\pi_*(\Lambda X) \otimes \mathbb{Q}$ has a natural graded Lie algebra structure. This can be recovered from a minimal model by taking Hochschild cohomology.
• Serre’s conjecture (theorem proved by Quillen and someone else independently) that every locally free sheaf over an affine space is in fact free.
• Bertini’s theorem on general members of linear systems
• Jordan-Holder theorem.
• Chevalley-Shephard-Todd theorem: The ring of invariants of a finite group acting on a complex vector space is a polynomial ring if and only if the group is generated by elements acting as pseudo-reflections.
• Hartman-Grobman theorem on linearizing flows
• Stable manifold theorem
• Poincare-Dulac normal form theorems for formal (and analytic) vector fields (and automorphisms).
• Atiyah-Patodi-Singer index theorem
• Kontsevich formula for number of rational curves in projective plane.
• The theorem on cohomology, i.e. the stronger form of Grauert’s theorem using the fact that if the cohomology jumps, it jumps in adjacent degrees - stronger form of Euler characteristic staying constant.
• Sikorav’s theorem about symplectomorphisms of $U \times T^n$.
• Different types of algebras can be seen as representations of operads.
• ojasiewicz inequality
• Taking incidence-correspondances seriously in algebraic geometry
• Uniformization theorem (all proofs are hard, and I have yet to read one). This for example implies that the mapping class group of the sphere is trivial.
• Kirwan’s convexity for non-abelian compact Lie group actions on symplectic manifolds.
• Homotopy transfer theorem has a generalization to all operads on chain complexes. This uses the notion of a free operad, which explains what is $A_\infty$, $L_\infty$, etc... The proof (given by Markl) uses colored operads in an essential way.
• Formality of the $E_n$ operad and Kontsevich formality can both be done over the rational numbers.
• Vassiliev’s finite type knot invariants, and configuration space integrals that produce them.
• There are operations on the homology of infinite loop spaces (called Dyer-Lashef operations) similar to Steenrod squares. Some of these also persist in $E_n$-spaces.
• Flux conjecture in symplectic geometry
• Goodwillie calculus
• Universal Novikov homology is an enhanced version of circle valued Morse theory. Enhanced in a sense very similar to Family Floer homology.
• There is a nice cell structure on the (one point compactification of) unordered configuration space.
• Laurent expansions of holomorphic functions in several complex variables converge in domains which are preimages of convex domains in the Log plane. In the tropical limit these become convex polygons.
• To see where the corner locus being the zero set in tropical geometry comes from see Viro’s dequantization paper. Maslov’s dequantization of the operation also explains some things (especially in the real case).
• Two-level principle of Grothendieck. One manifestation of this seen in pair-of-pants decompositions.
• DG-enhancements come in handy if one wants to produce models from not necessarily strong exceptional collections.