1. Lecture 8, the second half

1.1. Gauge theory basics. Let $G \to P \to X$ be a principal $G$-bundle. $G$ here will mostly be a compact Lie group. Our convention is that $G$ acts on the right.

A connection on $P$ is a choice of complements to the vertical subbundle of $TP$, i.e. a horizontal subbundle. What this is supposed to do for us is to define a notion of path lifting/parallel transport. We want to think of $P$ as being homogenous in the fiber directions (frame bundles are really what we have in our minds), so we also require the obvious compatibility with the group action for the path lifting.

Choosing a complement to a given subspace is equivalent to choosing a projection to that subspace. This is convenient because we have an isomorphism of vector bundles $P \times g \to V_{TP}$ given by the infinitesimal action of $G$. To make this map equivariant we are forced to use the (right) adjoint action of $G$ on $g$.

Putting all this together, we define a connection on $P$ to be a $g$-valued one-form $TP \to g$, which is identity when restricted to the vertical bundle, and equivariant with respect to the adjoint action on $g$. We denote the set of all connections by $\mathcal{A}_P$.

$\mathcal{A}_P$ is an affine space over $\Omega^1(M, \text{ad} P)$, where $\text{ad} P := P \times_{\text{Ad}} g$. Let us explain this a little bit. If $A, A' \in \mathcal{A}_P$, then $\theta = A - A'$ is a horizontal $g$-valued one form with the equivariance condition $R_g^* \theta = \text{Ad}(g^{-1}) \circ \theta$, where $R_g$ is multiplication by $g$. Hence, if we choose a vector at a point $p$ in $X$, we get a well defined element of $\pi^{-1}(p) \times_{\text{Ad}} g$, as desired.

We call equivariant fiber preserving maps $P \to P$ gauge transformations. Let us denote the group of all gauge transformations by $G_P$. We want our constructions to be independent under this gauge group action - global symmetries of our system. As with $\text{ad} P$ above, we can think of the gauge transformations as sections of $\text{Ad}P := P \times_{\text{Ad}} G$ - fix point $p \in X$ get a well defined element of $\pi^{-1}(p) \times_{\text{Ad}} G$. $\text{Ad}P$ is a bundle of Lie groups, i.e. multiplication makes sense, but it is not a principal $G$-bundle. One should think of $\text{ad} P$ as the Lie algebra of $\text{Ad}P$ - more to come about this.

If we start with a $U(n)$ bundle $P$, and form the associated complex vector bundle $E$, which comes equipped with a Hermitian form, $\text{Ad}P$ and $\text{ad} P$ are naturally isomorphic to the bundle of unitary and skew Hermitian transformations of $E$.

Remark 1. There is some confusion in both mathematics and physics literature about what is called a gauge transformation.

1.2. Symplectic nature of the space of connections on bundles over Riemann surfaces. We have been talking about symplectic geometry and moment maps, and those things make an appearance in this context as well. As a baby case, consider a $U(1)$-bundle over a closed Riemann surface $\Sigma$. $\mathcal{A}_P$ in this case is an affine space over $\Omega^1(\Sigma)$, hence tangent spaces are identified with $\Omega^1(\Sigma)$. This space (as a manifold) has a natural symplectic form given by $\omega(\alpha, \beta) = \int_{\Sigma} \alpha \wedge \beta$.

What should nondegeneracy mean in this infinite dimensional context? One possible definition would be to ask for a compatible almost complex structure (instead of a Riemannian metric we want an inner product as in Hilbert spaces at each tangent space). Hodge star acting on one forms gives such an almost complex structure:

\[
\int_{\Sigma} \alpha \wedge \star \alpha > 0.
\]
Note that Hodge star acting on one forms is really equivalent to the almost complex structure of $\Sigma$, i.e. depends only on the conformal structure and orientation, no need for a metric.

**Remark 2.** $\star^2 = (-1)^{p(n-p)}$ on $\Omega^n(X^n)$.

2. Lecture 9

2.1. Symplectic nature of the space of connections ctd.

Recall that we had described the structure of an “infinite dimensional symplectic manifold” on $A_P$ where $P$ is a principal $U(1)$-bundle over $\Sigma$, a closed Riemann surface. Now we add to that a Hamiltonian action of $\Omega^0(\Sigma)$ given by $f \cdot A = A + df$. Since the symplectic form on $A_P$ is translation invariant this is a symplectic action. Let us show that it is sensible to call this a Hamiltonian action by computing a moment map for it.

$\Omega^0(\Sigma)$ is a vector space so its Lie algebra is itself. We take a leap of faith and think of the dual of $\Omega^0(\Sigma)$ as $\Omega^2(\Sigma)$, where the pairing is given by integration. Hence the moment map will be a map $\mu : A_P \to \Omega^2(\Sigma)$. Fixing a base point $A_0$, we can identify $A_P$ with $\Omega^1(\Sigma)$, and we feel eligible to make the wild guess that $\mu$ could be something like the exterior derivative.

It is evident that the map $\text{Lie}(\Omega^0(\Sigma)) = \Omega^0(\Sigma) \to \text{Vect}(A_P)$ is given by $\xi = f \mapsto df = \hat{\xi}$, where $df$ is seen as a (constant) vector field on $A_P$. Formally this should satisfy for every $A \in A_P$, $\alpha \in \Omega^1(\Sigma)$,

$$\omega_A(\alpha, \hat{\xi}_A) = \alpha(\mu(\cdot), \xi >),$$

where the RHS is the derivative of function $\mu(\cdot), \xi >$ in the $\alpha$ direction. Now let us check that $\mu = d$ satisfies this equation.

$$(1) \quad \omega_A(\alpha, \hat{\xi}_A) = \int_\Sigma \alpha \wedge df.$$ 

$$(2) \quad \omega_A(\alpha, \hat{\xi}_A) = \int_\Sigma \alpha \wedge df.$$ 

$$(3) \quad \alpha(\mu(\cdot), \xi >) = \lim_{t \to 0} \int_\Sigma \frac{fd(A - A_0 + ta) - \int fd(A - A_0)}{t} = \int_\Sigma fda,$$

which finishes the proof by Stokes theorem and Leibniz rule.

**Remark 3.** Choosing a base point was not very pleasant. A more slick way of describing this moment map is to say that it is the curvature of the connection. We will talk more about this.

2.2. Covariant derivative. Let $E$ be an associated vector bundle. Then parallel transport in $P$ defines a notion of parallel transport in $E$. This gives us a way to covariantly differentiate sections of $E$, which is equivalent to a map:

$$(4) \quad d_A : \Omega^0(X, E) \to \Omega^1(X, E).$$

This satisfies the Leibniz rule $d_A(fs) = df \cdot s + f d_A s$. We can extend $d_A$ to a map $d_A : \Omega^*(X, E) \to \Omega^{*+1}(X, E)$ by requiring the more general Leibniz rule with forms:

$$(5) \quad d_A(\theta \wedge s) = d\theta \wedge s + (-1)^{d\theta}d_A s,$$

for any $\theta \in \Omega^k(X)$ and $s \in \Omega^m(X, E)$, $k, m \in \mathbb{Z}$.

We can describe these operations using the bundle $ad P$ - without going to the associated vector bundle. By taking the derivative of the system of parallel transports we get a system of parallel transports on $ad P$, which defines a covariant
derivative $d_A : \Omega^0(X, \text{ad } P) \to \Omega^1(X, \text{ad } P)$. We extend this to $d_A : \Omega^*(X, \text{ad } P) \to \Omega^*(X, \text{ad } P)$ as before.

2.3. **Further structures on ad $P$.**

- The Lie algebra structure of $\mathfrak{g}$ defines a Lie algebra structure on the fibers of $\text{ad } P \to X$ since the adjoint representation $G \to \mathfrak{gl}(\mathfrak{g})$ respects the Lie bracket. In particular, $\Omega^0(X, \text{ad } P)$ is naturally a Lie algebra.
- Using the wedge product of forms, we can extend this to a graded Lie algebra structure on $\Omega^*(X, \text{ad } P)$:

$$[\cdot \wedge \cdot] : \Omega^*(X, \text{ad } P) \times \Omega^*(X, \text{ad } P) \to \Omega^*(X, \text{ad } P).$$

Note that a graded Lie algebra is a graded vector space with a bracket satisfying the super analogues of skew-symmetry $[a, b] = -(-1)^{|a||b|}[b, a]$ and the Jacobi identity:

$$(-1)^{|a||c|}[a, [b, c]] + (-1)^{|b||a|}[b, [c, a]] + (-1)^{|b||c|}[c, [a, b]] = 0.$$
- If we are given an ad-invariant bilinear form on $\mathfrak{g}$, we also get a map:

$$\cdot \wedge : \Omega^*(X, \text{ad } P) \times \Omega^*(X, \text{ad } P) \to \Omega^*(X),$$

defined exactly in the same manner - using the bilinear form instead of the bracket. This satisfies $a \wedge b = (-1)^{|a||b|}b \wedge a$.
- If we in addition choose a Riemannian metric on $X$, we can define a symmetric bilinear form on $\Omega^*(X, \text{ad } P)$ by extending the Hodge star trivially. If the ad-invariant bilinear form is positive definite, then this is an $L^2$-inner product.
- One can check that $d_A$ is a (super)-derivation on $\Omega^*(X, \text{ad } P)$, and satisfies the only super Leibniz identity that makes sense for the $\cdot \wedge \cdot$ operation.

2.4. **Gauge group action on connections.** Let $A \in \mathcal{A}_P$, and $g \in \mathcal{G}_P$. We can simply pull back (or push forward, but pulling back is more natural from a differential form point of view) the horizontal subspaces to define a new connection $g \cdot A$. We want a formula for this action: $A - g \cdot A$ is an ad $P$ valued one form, what is it? The answer is $g(d_A g^{-1})$. To make sense of this expression choose a faithful representation of $G$, and get $\text{Ad } P \subset \text{End}(E) = E \otimes E^*$. Covariant derivative can be extended to $E \otimes E^*$ and this is how we make sense of $d_A g^{-1}$. Hence, when we take a vector in $T_p X$, $g \circ (d_A g^{-1}(v))$ as an element of $\text{End}(E_p)$ actually lies in the subspace $\text{ad } P_p \subset \text{End}(E_p)$.

This is a consequence of the Leibniz rule. Namely if $v \in \Gamma(E)$, $d_{g \cdot A} v = g(d_A(g^{-1}v))$, where $g$ is seen as an element of $\Gamma(\text{End}(E))$ - this is just what it should be up to where the inverse is. Leibniz rule then tells $d_{g \cdot A} v = d_A v + g_\ast(d_A g^{-1})((v)$.

This is related to the canonical $\mathfrak{g}$-valued one form on a Lie group $G \subset \text{Gl}(V)$. This form $\theta$ is given by $g^{-1}dg$. This is a symbolic and totally misleading expression which means you write $g = (x_{ij})$ matrix of variables then multiply matrices $g^{-1}$ and $(dx_{ij})$ symbolically and get a matrix of one forms.

One can derive some expression without choosing a representation but it didn’t seem very useful to me. If one understands the connection with the canonical one form, maybe one can write down a formula. If one tries to glue a connection from local pieces the canonical one form comes up naturally - what is the connection one form associated to the trivial connection on a trivial $G$-bundle? This is explained in Kobayashi-Nomizu.
2.5. Curvature of a connection. We have advocated the point of view that a connection is a distribution satisfying certain properties. The first question one asks about a distribution is whether it is integrable or not. By Frobenius theorem, this is equivalent to $A([X,Y])$ vanishing for all horizontal vector fields $X,Y$. Since we have a means of projecting any vector field to a horizontal one we define:

$$F_A(X,Y) = A([X_h,Y_h]) \in \Gamma(P, g),$$

for all vector fields $X,Y$ on $P$. A simple consequence of the Leibniz rule for the Lie bracket is that $F_A$ is tensorial, i.e. commutes with the action of functions. Note that this holds in general for any distribution - if a complement is not given project to the quotient. Moreover $F_A$ is horizontal and equivariant in the right way. Hence $F_A \in \Omega^2(X, \text{ad} P)$.

Here I want to make another digression to yet another way to describe what is a connection and its curvature. Given $G \to P \to X$ we already talked about the two naturally defined vector bundles over $X$: the tangent bundle $TX$, and the bundle of $G$-equivariant vertical vector fields, which is isomorphic to $\text{ad} P$ of course. But, we also have the bundle of $G$-equivariant vector fields, which we call $E$ now. These three sit in a short exact sequence,

$$0 \to \text{ad} P \to E \to TX \to 0$$

In other words we also have a given extension of these two fundamental vector bundles. A connection is simply a splitting of this SES as vector bundles- equivariance is built into it.

Sections of these vector bundles also come with Lie algebra structures (by virtue of being vector fields of some sort, the two notions coincide for $\text{ad} P$), and the sequence is an exact sequence of Lie algebras when we take global sections. Curvature is then a measure of how much the connection fails to be a splitting of SES of Lie algebras - see Atiyah-Bott for details.

**Remark 4.** If you look at the definition of Atiyah class, the whole thing boils down to writing down a linear connection as a splitting of some exact sequence - and in the holomorphic or algebraic cases a splitting may not exist, i.e. the extension may not be trivial.

Another way to understand what is a curvature is to look at

$$\Omega^0(X, \text{ad} P) \xrightarrow{d_A} \Omega^1(X, \text{ad} P) \xrightarrow{d_A} \Omega^2(X, \text{ad} P) \cdots$$

Maybe you had hoped before that this would be a chain complex, but it is not, and curvature is also exactly the measure of this failure. The standard computation shows that $d_A^2 : \Omega^0(X, \text{ad} P) \to \Omega^2(X, \text{ad} P)$ is tensorial. The same magic of signs shows that $d_A^2$ is a derivation when we feed in two vectors, hence we get that $d_A^2 = [F_A \wedge \cdot]$ for some $F_A \in \Omega^2(X, \text{ad} P)$ - here I used semisimplicity of $g$ but maybe it is not needed.

So we now have two descriptions of the curvature of a connection. We can compute that in both cases the following formula holds:

$$F_A = dA + \frac{1}{2}[A \wedge A]$$

as a $g$ valued two form on $P$. For the first case (integrability obstruction) one needs to use the Cartan formula for exterior differentiation (a.k.a. the curvature identity),...
and the second one \((d_A \text{ is not a differential})\) is a local computation, in fact from the same computation (now can be done globally) we get the more general:

\begin{align*}
F_{A+a} &= F_A + d_Aa + \frac{1}{2}[a \wedge a] \\
\end{align*}

**Remark 5.** The classical Maurer-Cartan equation is just a very special case of the structure equation (12) above.

3. **Lecture 10**

3.1. **Curvature of a principal bundle over a Riemann surface as a moment map.** We have \(G \to P \to \Sigma\), where \(\Sigma\) is a closed Riemann surface. \(A_P\) is an affine space over \(\Omega^1(\Sigma, \text{ad} P)\). Fixing an ad-invariant positive definite bilinear form on \(g\), we get a non-degenerate (in the sense discussed in lecture 9) symplectic structure on \(A_P\):

\begin{align*}
\omega_A(\alpha, \beta) &= \int_\Sigma \alpha \wedge \beta.
\end{align*}

We are interested in analyzing \(G_P\) acting on \(A_P\) keeping in mind the symplectic structure on \(A_P\).

**Proposition 1.** \(G_P\) acts symplectomorphically on \(A_P\).

**Proof.** We compute the map induced on tangent spaces. Note the following computation is done in \(\text{End}(E)\) for a faithful representation.

\begin{align*}
g \cdot (A + ta) &= g \cdot A + t(a + g[a \wedge g^{-1}]) = g \cdot A + tgag^{-1}.
\end{align*}

This finishes the proof since we defined \(\omega\) by an ad-invariant form. \(\square\)

As we mentioned before \(\Omega^0(\Sigma, \text{ad} P)\) is the Lie algebra of \(G_P\) - in particular we can exponentiate elements of the former to get one parameter subgroups of the latter. Considering \(\Omega^2(\Sigma, \text{ad} P)\) to be the dual of \(\Omega^0(\Sigma, \text{ad} P)\) using the integration pairing, we have the following lovely proposition - if we want to be more precise we can say \(\Omega^2(\Sigma, \text{ad} P)\) embeds into the dual and compose with that the map below. If you go back to lecture 9, you will see that this is precisely the same computation with the \(U(1)\) case - there some steps were too obvious to be recognized as steps.

**Proposition 2.** \(A_P \to \Omega^2(\Sigma, \text{ad} P)\) given by \(A \mapsto F_A\) is a moment map for the gauge group action.

**Proof.** The only extra step here is to determine \(\Omega^0(\Sigma, \text{ad} P) \to \text{Vect}(A_P)\). Let \(s \in \Omega^0(\Sigma, \text{ad} P)\). We want to compute

\begin{align*}
\frac{d}{dt} \bigg|_{t=0} \exp ts \cdot A \in T_A A_P &= \Omega^1(\Sigma, \text{ad} P).
\end{align*}

Using Leibniz rule we get

\begin{align*}
\exp ts \cdot A &= A - tdAs \exp(st) \exp(-st) = A - tdAs.
\end{align*}

Hence the answer is \(\dot{s} = -dAs\). Now we want to check that the equation

\begin{align*}
\alpha(<\mu(\cdot, s >) &= \omega(\dot{s}, \alpha)
\end{align*}
holds for \( \mu(A) = F_A \). The right hand side is
\[
- \int_\Sigma d_A s \wedge \alpha.
\]
Left hand side is equal to:
\[
(6) \quad \frac{d}{dt} \bigg|_{t=0} \int_\Sigma F_{A+ta} \wedge s = \frac{d}{dt} \bigg|_{t=0} t \int d_A s \wedge s + O(t^2) = \int_\Sigma d_A a \wedge s.
\]
The appropriate Leibniz rule (sign!) plus the Stokes theorem finishes the proof.

\[\square\]

3.2. Pushing the finite dimensional analogy further: representation variety as symplectic reduction. If we were in the finite dimensional context, then we would have a symplectic structure on \( \mu^{-1}(0)/G \) - given that the \( G \) action on \( \mu^{-1}(0) \) is free say. This still goes through in our case, but we don’t have the required technology yet - we have to work in spaces where we have an inverse function theorem to get a smooth structure on \( \mu^{-1}(0) \) - then the symplectic structure will work out nicely. Of course we may get an orbifold. We will actually do this at some point, but not now. We now switch gears and describe \( AP//G \) from the holonomy point of view.

3.3. Holonomy and representation variety. Given a connection \( A \) on \( G \to P \to X \) parallel transport describes for us a group homomorphism \( \text{hol}_A : \Omega(X, x_0) \to G \). This is called the holonomy of the connection.

The situation is much nicer when \( A \) is a flat connection, i.e. \( F_A = 0 \). In this case, the horizontal distribution is integrable. Take a lift \( p_0 \) of \( x_0 \) to \( P \). \( p_0 \) belongs to a unique leaf \( L \) of the foliation, and \( L \) and \( \pi^{-1}(x_0) \) are transversely intersecting at \( p_0 \). Since the lifts of loops based at \( x_0 \) which lift to \( p_0 \) will stay in \( L \), this shows that a small, contractible loop maps to identity under the holonomy representation. This (by chopping up the homotopy to small pieces) shows that homotopic loops map to the same element under the holonomy representation. Hence, once we choose a point \( p_0 \) above \( x_0 \) we have:
\[
(7) \quad \text{hol}_A : \pi_1(X, x_0) \to G.
\]
Changing the point \( p_0 \) results in an overall conjugation by an element of \( G \). Hence the holonomy map is meaningful only up to overall conjugation. Notice that the gauge group action also reduces to this same conjugation action.

Conversely, if we are given a map \( \rho : \pi_1(X, x_0) \to G \) we can construct a principal \( G \)-bundle \( P_\rho \) with a flat connection by the formula \( \hat{X} \times_{\pi_1(X, x_0)} G \), where \( \hat{X} \) is the universal cover of \( X \) - connection is obtained from the trivial foliation on \( \hat{X} \times G \). Hence we have the following natural bijection:
\[
(8) \quad \{ A \in AP \mid F_A = 0 \}/\text{gauge} \simeq \{ \rho : \pi_1(X, x_0) \to G \mid P_\rho \simeq P \}/\text{conjugation}
\]

Remark 6. Note that people are usually careless about the condition on the RHS (including this note, but should be clear from context), but it is necessary. Not all flat bundles are trivial - all real vector bundles have flat connections but they may not be trivial according to their orientability, a.k.a first Stiefel-Whitney class.

We define: \( R_G(X) := \{ \rho : \pi_1(X, x_0) \to G \}/\text{conjugation} \) and \( R_G(X, P) = \{ \rho : \pi_1(X, x_0) \to G \mid P_\rho \simeq P \}/\text{conjugation} \). Clearly,
\[
(9) \quad R_G(X) = \coprod R_G(X, P).
\]
3.4. Representation varieties for Riemann surfaces. Recall the standard representation of the fundamental group of a closed Riemann surface of genus $g$:

(10) \[ \{a_1, b_1, \ldots, a_g, b_g \mid \prod[a_i, b_i] = 1 \}. \]

For example, let us think about

(11) \[ R_{SU(n)}(\Sigma_g) = \{A, B \in SU(n) \mid AB = BA\}/\text{conj}. \]

For $\Sigma = T^2$, using the spectral theorem and the fact that two commuting diagonalizable matrices are simultaneously diagonalizable, we see that $R_{SU(n)}(T^2) = T \times T/W$, where $T$ is a maximal torus inside $SU(n)$, for example the diagonal matrices, and $W$ is the Weyl group $\text{Normalizer}(T)/T = S_n$. For $n = 2$, this is $S^1 \times S^1/\mathbb{Z}_2$, where the nontrivial element in $\mathbb{Z}_2$ acts by complex conjugation if we see $S^1 \times S^1 \subset \mathbb{C}^2$ in the standard way. One can see that this is topologically a 2-sphere (it looks like a pillowcase, when you represent the torus as a rectangle with identifications), and the more general result

(12) \[ R_{SU(n)}(T^2) \simeq \mathbb{P}^{n-1} \]

was stated in class - even more generally, for $G$ compact and simple group one gets weighted projective spaces apparently. Note that the representation variety has a natural Kahler structure (reduction can be upgraded to the Kahler case too), so supposedly these isomorphisms are of Kahler manifolds (orbifold), but I am a bit confused because it seems like in the $SU(2)$ case what we get is not the ordinary $\mathbb{P}^1$ but has a nontrivial orbifold structure.

Remark 7. There is a theorem that says if a finite group $G$ acts (smoothly, no other conditions) on a closed manifold $X$, then the rational (it is enough that the order of the group is invertible in the coefficient ring) cohomology of $X/G$ is isomorphic to $H^*(X)^G$ (for proof see Bredon - Intro. to Compact Transformation Groups). We can try to prove (12) at the level of cohomology using this:

(13) \[ H^*(T \times T, \mathbb{Q}) = \Lambda^*(x_1, \ldots, x_n)/(\Sigma x_i = 0) \otimes \Lambda^*(y_1, \ldots, y_n)/(\Sigma y_i = 0), \]

where $x$ and $y$'s are of degree 1, and the $S_n$ action permutes $x$ and $y$ variables simultaneously. It should be that the ring of invariants is generated by $x_1 y_1 + \ldots + x_n y_n$.

Remark 8. What Atiyah-Bott paper does (of course among other things) is to compute the Poincare polynomials, i.e. Betti numbers, of representation varieties of Riemann surfaces.

4. Lecture 11

4.1. An interesting oriented flat real vector bundle over a torus. One reason we got such nice explicit results for $SU(n)$ was the simultaneous diagonalizability property. This result generalizes to all simply connected compact Lie groups (this is proven by Borel in the paper Sous-groupes commutatifs et torsion des groupes de Lie compacts) in the only possible way. Let us consider $SO(3)$ and see the kind of complications that can arise in the non-simply-connected case.

Inside $SO(3)$, there are two maximal abelian subgroups $SO(2)$, the maximal tori, and also the Klein group, embedded for example as the diagonal matrices with entries $\pm 1$. Let us denote the diagonal Klein group by $V_4 \subset SO(3)$. Let $x, y$ be two nontrivial elements of $V_4$. We can show easily that $x$ and $y$ can not be
simultaneously conjugated to lie inside a maximal torus - if there is such conjugation it puts $xy$ in the same circle as well, now look at how many elements can have trace $-1$ in a maximal circle.

Hence, the flat principal $SO(3)$-bundle $P_{\rho}$ corresponding to one of the surjective maps $\rho : \pi_1(T^2) \to V_4 \subset SO(3)$ is an interesting one. For the associated vector bundle

$$E := P_{\rho} \times_{SO(3)} \mathbb{R}^3 \simeq T^2 \times_{\pi_1(T^2)} \mathbb{R}^3 \simeq V_{(0,1)} \oplus V_{(1,0)} \oplus V_{(1,1)},$$

where $V_{(p,q)}$, for $p, q$ coprime integers, is the real line bundle over $T^2$, which is obtained by pulling back the Moebius line bundle over $S^1$ via the map induced by $\mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto px + qy$. Let $\alpha$ and $\beta$ be the generators of $H^1(T^2, \mathbb{Z}_2)$ corresponding to the $x$ and $y$ coordinates. Then the Stiefel-Whitney class $w(V_{(p,q)})$ is easily computed to be (by naturality) $p\alpha + q\beta$. By Cartan’s formula, the total Stiefel-Whitney class $w(E) = 1 + \alpha \wedge \beta$. So this is a nontrivial flat vector bundle, which is nontrivial for nontrivial reasons.

4.2. Some locally final thoughts on representation varieties. The last example should be understood as an indication that the representation variety can be complicated. Let us say a bit more on this.

Here is a homework exercise:

- Consider the commutator map $SU(2) \times SU(2) \to SU(2)$, $(A, B) \to ABA^{-1}B^{-1}$. Show that unless $A$ and $B$ commute, the differential at $(A, B)$ is surjective. (sucks for representation variety)
- What is the preimage of $-I$? Hint: Show that every such pair simultaneously conjugate to $(i, j)$ under the identification of $SU(2)$ with unit quaternions.

Another related question is: what are the possible stabilizers of representations under conjugacy action? An equivalent question is: what are the possible centralizers of of subgroups? Let me just note the abstract isomorphism types here and ignore how they are embedded (there are choices there). For $SU(2)$, the list is quite simple: $\mathbb{Z}_2 \subset U(1) \subset SU(2)$. For $SO(3)$, it gets a little bit more complicated:

$$1 \subset \mathbb{Z}_2 \subset V_4, SO(2) \subset O(2) \subset SO(3).$$

4.3. Projectively flat connections. We have been talking about $SU(n)$, but let’s briefly consider what happens if we instead talk about $U(n)$, which is what Atiyah-Bott does. We clearly have the splitting:

$$ad P \simeq ad P \times_{U(n)} \mathfrak{su}(n) \oplus \mathbb{R}.$$

If we now run the moment map argument as in the $SU(n)$ case, we will find that

$$\mu(A) = F_A - i\omega,$$

where $\omega \in \Omega^2(\Sigma, ad P)$, but can only take values in the center of $\mathfrak{u}(n)$, in other words, $\omega$ is an ordinary two form.

Now we want to do symplectic reduction, so we consider the set $\{\mu(A) = 0\}$. As our final word, notice that this set is empty unless $\omega$ has the integrality condition:

$$[-n\omega] \in 2\pi H^2(\Sigma, \mathbb{R}),$$

This is because of the Chern-Weil formula $\frac{1}{2\pi} \text{tr} F_A = c_1(P)$, and that Chern classes are integral.
5. Lecture 12

5.1. Characteristic classes. A characteristic class is an assignment to each (real, complex, or quaternionic) vector bundle $E \to X$ a cohomology class $c(E)$ such that that $f^*(c(E)) = c(f^*(E))$, for any $E \to X$ and $f : Y \to X$. It is clear how to extend this definition to principal $G$-bundles.

A classifying space $BG$ for a group $G$ is a topological space with a given principal $G$-bundle $EG \to BG$ such that:

- for every principal $G$-bundle $P \to X$, there exists a map $f : X \to BG$ such that $f^*EG \simeq P$,
- for any two $f,f' : X \to BG$, $f$ is homotopic to $f'$ if and only if $f^*EG \simeq f'^*EG$.

Classifying spaces exist, so its defining property shows that a characteristic class is always obtained by pulling back classes from the cohomology of $BG$. Note that the homotopy equivalence class of $BG$ is well defined.

In the case of vector bundles (i.e. $G = O(n), U(n)$, or $Sp(n)$, here $Sp(n)$ is the quaternionic unitary group) we can construct a classifying space as the infinite Grassmannian of $n$-planes $Gr_n(\mathbb{F})$, where $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, with the tautological $n$-plane bundle (the corresponding frame bundle is the infinite Stiefel manifold) over it. It can be shown directly in this case that we have a classifying space (for nice spaces) by construting maps to $\mathbb{F}^\infty$.

In general, $EG \to BG$ is a classifying space if and only if $EG$ is contractible. A construction using what is called a join of two topological spaces was given by Milnor. Namely, if $X$ and $Y$ are topological spaces, $X \star Y$ is the space formed by abstract paths between $X$ and $Y$,

\[ X \star Y := X \times Y \times I/(x,y,0) \sim (x', y, 0), (x', y, 1) \sim (x, y, 1). \]

Then, Milnor constructs $EG$ as the infinite join $G \star G \star \ldots$ with the natural $G$ action, and $BG$ as $EG/G$. If $G = \mathbb{Z}_2$ for example, it can be seen explicitly that we get $S^\infty$ with the antipodal action.

5.2. Stiefel-Whitney classes. Note that for $O(n)$, we have $H^*(BO(n), \mathbb{Z}_2) \simeq \mathbb{Z}_2[w_1, \ldots, w_n]$, where $w_i$ has degree $i$.

Remark 9. This can be seen by understanding the Schubert cells of $Gr_n(\mathbb{R}^\infty)$. At least for the finite dimensional approximations $Gr_n(\mathbb{R}^k)$, we can see this decomposition using the flow of the Morse function $\Sigma x_i^2$ on $\mathbb{R}^k$ - this defines a flow on $Gr_n(\mathbb{R}^k)$. Is there a Morse function on $\mathbb{R}P^{n-1}$ inducing this flow?

We define the Stiefel-Whitney classes by pulling back $w_i$'s. Some things you may have heard before:

- $w_1$ is the obstruction to orientability, in other words reducing the structure group from $O(n)$ to $SO(n)$.
- For an orientable bundle, i.e. $w_1 = 0$, $w_2$ is the obstruction to the existence of a spin structure, in other words reducing the structure group from $SO(n)$ to $Spin(n)$.

Note that another way to phrase these existence questions is by asking for a lift of the classifying map to $BH \to BG$.

One way to interpret these classes is as obstructions to having linearly independent sections over skeleta. Let us fix a CW-complex structure on our manifold.
• Assume that we are given a frame over the 0-skeleton and we want to extend it to the 1-skeleton. Let us try to do this one by one for each cell. Over a cell $I \rightarrow X$ we can trivialize the vector bundle. We want to extend the given bases at the endpoints to the interior, and clearly we can do this if and only if the two bases have the same orientation with respect to our trivialization. A convenient way to think about this as assigning 0 or 1 to each one cell, in other words a cellular one cochain. Changing the given frames on the 0-skeleton changes this cochain by coboundaries. Hence, we have a well defined cellular $\mathbb{Z}_2$ cohomology class, of which vanishing is equivalent to the existence of some choice of frames over the 0-skeleton, which can be extended to the 1-skeleton. This cohomology class is of course the first Stiefel-Whitney class.

• How about having a nonzero section over all of $X$? One can start with an arbitrary section that is transverse to the zero section. The zero set of this section is a submanifold $Z$, and the homology class of this submanifold $[Z]$ is independent of chosen section as long as it is transverse. Unless this homology class is trivial, we can’t have a nonzero section. The $\mathbb{Z}_2$ Poincare dual of $[Z]$ coincides with $w_n$.

• In general $w_i$ is the obstruction to having $rk(E) - i + 1$ linearly independent sections over $i$-skeleton. There are two main points here:

(1) The fact the Stiefel manifold $V_k(\mathbb{R}^n)$ is $(n-k)$-connected and $\pi_{n-k+1}$ is infinite cyclic.

(2) The definition of cohomology with local coefficients and the notion of an obstruction to extending a section to one higher skeleton as a certain cohomology class. The local coefficient system can be twisted but its mod-2 reduction is not, and those are the SW classes.

5.3. Thom isomorphism. Let $\pi : E \rightarrow X$ be a rank $n$ real vector bundle. We denote by $E_0$ the complement of the zero section in $E$. All cohomologies here have $\mathbb{Z}_2$ coefficients, one can do $\mathbb{Z}$ by assuming orientability. Thom isomorphism tells us that there is a class, called the Thom class, $\tau \in H^n(E,E_0)$, such that the map $H^*(B) \rightarrow H^{*+n}(E,E_0)$ given by $\alpha \mapsto \pi^*(\alpha) \cup \tau$ is an isomorphism. It is recommended to the reader to keep track of the domains of the cup products in this discussion.

There is a canonical class $\tau \cup \tau \in H^{2n}(E, E_0)$, which we transfer to $H^n(B)$ using the Thom isomorphism. This class is the same as $w_n(E)$. Let us demonstrate this. Choose a generic section $s : X \rightarrow E$, which we can use to define $\hat{s} : (E, E_0) \rightarrow (E, E - \text{image of } s)$ via translation by $s$ in each section. One can check that:

(2) \[ \tau \cup \tau = \tau \cup \hat{s}^*\tau, \]

say after pulling back everything to $E$.

Now comes the heuristic part that can be made precise (I leave it as it was in the lecture). $\tau$ can be thought of as supported very near the zero section, and therefore $\hat{s}^*\tau$ is supported very near the image of $s$. What the cup product should do is one way or the other to get a class that is supported near their intersection, and this shows that the class we defined in $H^n(B)$ is very related to the zero set of $s$. 

6.1. Chern-Weil theory. We have been pulling classes back from the cohomology of $BG$ to get characteristic classes, so what is the cohomology of $BG$? This may seem like a very hard question, but in fact rationally it has a very concrete description for any compact Lie group:

$$H^*(BG, \mathbb{Q}) \simeq (\text{Polynomials on } t)^W,$$

where $t$ is the Lie algebra of a maximal torus in $G$, and $W$ is the Weyl group - superscript denotes invariants. Also note that the generators of the polynomial ring above have degree 2, not 1. In the $U(n)$ case, this means that the cohomology ring is isomorphic to the ring of symmetric polynomials in $n$ variables.

Chern-Weil theory gives a way to realize the isomorphism (1) above, by constructing a map \((\text{Polynomials on } t)^W \to H^*(BG, \mathbb{Q})\), where we think of $H^*(BG, \mathbb{Q})$ as all possible characteristic classes of principal $G$-bundles (see the definition of a characteristic class in the beginning of Lecture 12).

First, notice that we can think of the Weyl group invariant polynomials on $t$ as $\text{ad}$-invariant polynomials in the whole of $g$. Let us take such a polynomial $\tilde{f} : g \to \mathbb{R}$. We also assume that $\tilde{f}$ is homogenous of degree $k$. Also define for notational convenience $f(\xi) := \tilde{f}(\xi, \ldots, \xi)$.

We now want to define a cohomology class using this, for a given principal $G$-bundle $P \to X$. We could take a connection $A$, of which curvature is $F_A \in \Omega^2(X, \text{ad} P)$, and we could evaluate $f$ at $F_A$ to get a $2k$ differential form:

$$f(F_A) \in \Omega^{2k}(X)$$

Why we should do this is still a bit of a mystery to me (there is a paper of Bott that could help me) but it all fits very nicely together.

Bianchi identity implies that this form is closed. We need to show that the cohomology class of $f(F_A)$ is independent of $A$. For this it is an exercise to show that

$$\left. \frac{d}{dt} \right|_{t=0} f(F_{A+ta}) = kf(\text{ad} F_A, \ldots, F_A).$$

(As a hint note that the LHS is equal to $k\tilde{f}(d_A, F_A, \ldots, F_A)$.)

Now go back to $G = U(n)$. We have already commented that the $\text{ad}$-invariant polynomials are symmetric polynomials in $n$-variables in this case. There are two natural set of generators of this ring:

- Elementary symmetric polynomials: $\sigma_k$
- Newton polynomials: $x_1^k + \ldots + x_n^k$

Elementary symmetric polynomials arise as the graded pieces of $\det(1 + \xi)$, whereas the Newton polynomials are of $\text{tr}(e^{\xi})$. Hence, we can define two sorts of characteristic classes

- $\det(1 + \frac{i}{2\pi} F_A)$ - multiplicative under $\oplus$
- $\text{tr}(e^{\frac{i}{2\pi} F_A})$ - additive under $\oplus$

Of course the classes in one set can be represented in terms of the other (Newton’s formulas). $2\pi$ and $i$’s are to match with the topological definitions (which have a certain integrality). Finally the first set is called Chern classes, and the second Chern character.
Let us consider the following principal $PSL(2,\mathbb{R})$-bundle over a Riemann surface $\Sigma$ of genus $g > 1$. $PSL(2,\mathbb{R})$ acts on the Poincare disc. The uniformization theorem tells us that there is a map $\pi_1(\Sigma) \to PSL(2,\mathbb{R})$ such that $\Sigma = \mathbb{D}^2/\pi_1(\Sigma)$. Define the bundle:

$$P = \mathbb{D}^2 \times_{\pi_1(\Sigma)} PSL(2,\mathbb{R}).$$  

$PSL(2,\mathbb{R})$ action restricts to an action on the boundary circle $S^1$, so we can also define an $S^1$-bundle (not a principal $G$-bundle) over $\Sigma$:

$$Q = P \times_{PSL(2,\mathbb{R})} S^1 = S^1 \times_{\pi_1(\Sigma)} \mathbb{D}^2.$$  

Now, by construction, $P$ is a flat bundle. If there were some Chern-Weil-type description of characteristic classes of principal $PSL(2,\mathbb{R})$-bundles these characteristic classes would all vanish for $P$. $PSL(2,\mathbb{R})$ has the homotopy type of a circle, and hence $BPSL(2,\mathbb{R})$ is a $K(\mathbb{Z},2)$, which classifies second cohomology. The classifying map $\Sigma \to BPSL(2,\mathbb{R})$ of $P$ induces the zero map on cohomology, and hence is nullhomotopic. Therefore $P$ would be trivial. Then $Q$ would be trivial as well.

Considering the second description of $Q$, we see that it comes with a codimension 1 foliation which is transverse to the $S^1$ fibers - analogous to a flat connection. It is a theorem of Milnor and Wood that in this case we have:

$$| \langle e(Q), [\Sigma] \rangle | \leq -\chi(\Sigma)$$

We note here that it is possible to define Euler class for oriented sphere bundles (the obstruction definition works).

Apparently our bundle gives an equality case for this inequality - I did not really think about how to prove it. Hence it is not trivial.