(1) Chapter 6, # 11.
(2) Chapter 6, # 18.
(3) Chapter 6, # 22.
(4) Chapter 6, # 31.
(5) Chapter 7, # 1.
(6) Chapter 7, # 7.
(7) Chapter 7, # 8.
(8) Chapter 7, # 20.

9–10 The following extension of Chapter 6, #8. Recall from class that a normed vector space over $F = \mathbb{R}$ or $\mathbb{C}$ is a vector space together with a norm function $V \to \mathbb{R}$, $v \mapsto |v|$, such that:

(i) $|v| \geq 0$ for all $v$, with equality if and only if $v = 0$;
(ii) $|\lambda v| = |\lambda||v|$ for all $\lambda \in F$;
(iii) Triangle inequality: $|v + w| \leq |v| + |w|$ for all $v, w \in V$.

Show that the following are equivalent structures:
(A) A finite-dimensional inner product space;
(B) A finite-dimensional normed vector space satisfying the parallelogram identity:

$$
|u + v|^2 + |u - v|^2 = 2(|u|^2 + |v|^2),
$$

specifically: given (A), the function $|v| := \sqrt{\langle v, v \rangle}$ defines a normed vector space; given (B), there is a unique inner product $\langle -,- \rangle$ such that $|v| = \sqrt{\langle v, v \rangle}$ for all $v$.

**Note:** For the implication (A) $\Rightarrow$ (B), it is enough to cite results from the book—you do not need to rewrite the proof (although since we didn’t do it in class, I would recommend reading it, and also read the last two slides from Lecture 15 that we didn’t get to in class).

**Hint for (B) $\Rightarrow$ (A):** To obtain $\langle u, v \rangle$, use Chapter 6, #6 and #7 (which were on Homework 7, # 9). Then, it remains to use the parallelogram identity, together with the three axioms of normed vector spaces, to prove that the conditions of inner product spaces are all satisfied: positivity, definiteness, additivity, homogeneity, and conjugate symmetry.

**Further hints:** In order to prove homogeneity, first prove additivity. To prove additivity, show first that $\langle u, v + w \rangle = 2\langle u/2, v \rangle + 2\langle u/2, w \rangle$ using only some number of applications of the parallelogram identity (and the identity (ii) above). Finally, to prove homogeneity from additivity, in the case $F = \mathbb{R}$, first show that $\langle \lambda u, v \rangle = \lambda \langle u, v \rangle$ when $\lambda$ is a positive integer and when $\lambda = -1$, and use this to deduce it for negative integers and rational numbers. Then, for arbitrary real $\lambda$, note that there is a sequence $\lambda_j$ of rationals which converges to $\lambda$. Then, prove and apply the following continuity
property:

\[ \lim_{\lambda_j \to \lambda} \|u + \lambda_j v\| = \|u + \lambda v\|, \forall u, v \in V, \]

which you can prove using the triangle inequality. This then implies homogeneity for arbitrary \( \lambda \in \mathbb{R} \).

For the case \( F = \mathbb{C} \), show homogeneity when \( \lambda \in \mathbb{Q} + \mathbb{Q}i \) and use the above continuity property (again, prove it using the triangle inequality) to deduce it for \( \lambda \in \mathbb{C} \). You will need to know that, for all \( \lambda \in \mathbb{C} \), there is a sequence \( p_j + iq_j \) for \( p_j, q_j \in \mathbb{Q} \) (for integers \( j \geq 1 \)) such that \( \lim_{j \to \infty} (p_j + iq_j) = \lambda \).

Final hint: To show homogeneity for \( \lambda \in \mathbb{Q} + \mathbb{Q}i \), you can use homogeneity for \( \mathbb{Q} \) together with \( \langle iu, v \rangle = i\langle u, v \rangle \). This final equality will follow immediately from the definition of \( \langle -, - \rangle \) from the norm function that you are using.