The first six problems demonstrate how to compute the eigenvalues of a square matrix (or linear transformation). In general, all that one can do is produce a polynomial (of degree equal to the size of the matrix) whose roots are the eigenvalues; in the case where the polynomial has low degree or is of a special form, we can find its roots, as will be the case below.

On the other hand, once one knows the eigenvalues of a square matrix $A$, computing the eigenvectors is a simple matter: for each eigenvalue $\lambda$, its eigenspace is $\text{null}(A - \lambda I)$, which can be computed using Gaussian elimination.

(1) Use Gaussian elimination for the matrix $A - xI$ (where $x$ is a variable) to compute the (complex) eigenvalues of the following matrix. Show your work!

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

A: Here we go:

$$\begin{pmatrix} 1 - x & 2 & 3 \\ 4 & 5 - x & 6 \\ 7 & 8 & 9 - x \end{pmatrix} \mapsto \begin{pmatrix} 1 - x & 2 & 3 \\ 4(1 - x) & (5 - x)(1 - x) & 6(1 - x) \\ 7(1 - x) & 8(1 - x) & (9 - x)(1 - x) \end{pmatrix}$$

$$\mapsto \begin{pmatrix} 1 - x & 2 & 3 \\ 0 & (5 - x)(1 - x) - 8 & 6(1 - x) - 12 \\ 0 & 8(1 - x) - 14 & (9 - x)(1 - x) - 21 \end{pmatrix}$$

$$\mapsto \begin{pmatrix} 1 - x & 2 & 3 \\ 0 & x^2 - 6x - 3 & -6x - 6 \\ 0 & (-8x - 6)(x^2 - 6x - 3) & (x^2 - 10x - 12)(x^2 - 6x - 3) \end{pmatrix}$$

$$\mapsto \begin{pmatrix} 1 - x & 2 & 3 \\ 0 & x^2 - 6x - 3 & -6x - 6 \\ 0 & 0 & (x^2 - 10x - 12)(x^2 - 6x - 3) + (8x + 6)(-6x - 6) \end{pmatrix}$$

$$= \begin{pmatrix} 1 - x & 2 & 3 \\ 0 & x^2 - 6x - 3 & -6x - 6 \\ 0 & 0 & x^4 - 16x^3 - 3x^2 + 18x \end{pmatrix}.$$
Then, taking the difference, we would also get $9x + 15 = 0$, so $x = -\frac{15}{9}$, but this is not a root of either of these quadratics (neither of which have rational roots). So the two cannot have any roots in common.

As a result, when either $x = 0$ or $x^2 - 15x - 18 = 0$, then the above shows that the original matrix $A - xI$ must be noninvertible. The roots of $x^2 - 15x - 18$ are distinct: they are $\frac{15 + \sqrt{297}}{2}$ and $\frac{15 - \sqrt{297}}{2}$. This yields three distinct real eigenvalues. Since $A$ can have at most three eigenvalues (as we pointed out in class, or see ), these must be all the eigenvalues, and nonzero eigenvectors for them form an eigenbasis of $A$, in which $A$ becomes a diagonal matrix.

In conclusion, the eigenvalues are $0$, $\frac{15 + \sqrt{297}}{2}$, and $\frac{15 - \sqrt{297}}{2}$.

**Remark 0.2.** By the way, we were able to know in advance that $1 - x$ would be a factor of the final quartic polynomial. This is because, in general, the product of all of these diagonal entries must be a multiple of the product of all polynomials we multiplied rows by. In this case, the product of all three polynomials must be a multiple of $(1 - x)^2(x^2 - 6x - 3)$, and hence the third polynomial must be a multiple of $1 - x$.

In general, if we take the product of the diagonal entries and divide by this, the resulting polynomial is (after possibly rescaling to get a monic polynomial) the characteristic polynomial, whose roots are exactly the eigenvalues, appearing the number of times equal to the number of times they appear on the diagonal when the matrix is conjugated to upper triangular form.

The reason for this is that the characteristic polynomial equals $\det(A - xI)$, and in general to compute $\det(B)$ for any square matrix $B$, one can apply Gaussian elimination [not necessarily requiring the pivot entries to be one], then multiply the diagonal entries of the final row echelon matrix and divide by everything that we multiplied the rows by as well as multiplying by $-1$ to the power of the number of times we swapped a pair of rows. This is a consequence of the formula $\det(BC) = \det(B)\det(C)$, together with the determinants of the matrices corresponding to row operations ($\lambda$ for the matrix which multiplying a single row by $\lambda$; $-1$ for the permutation matrix which swaps two rows; and 1 for the triangular matrix which ads a multiple of one row to another row.)

(2) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Mat}(2, 2, \mathbf{F})$ be a general $2 \times 2$-matrix. Define its **trace** and **determinant** by

$$\text{tr}(A) := a + d, \quad \det(A) = ad - bc.$$ 

Perform Gaussian elimination on $A - xI = \begin{pmatrix} a - x & b \\ c & d - x \end{pmatrix}$, where $x$ is a variable, to prove that the eigenvalues of $A$ are exactly the roots of the polynomial

$$x^2 - \text{tr}(A)x + \det(A).$$

A: We slightly modify Gaussian elimination, making care to only perform invertible operations (rescaling only by nonzero numbers). Let us first assume that $x \neq a$. Then we see that

$$\begin{pmatrix} a - x & b \\ c & d - x \end{pmatrix} \mapsto \begin{pmatrix} a - x & b \\ c(a - x) & (d - x)(a - x) \end{pmatrix} \mapsto \begin{pmatrix} a - x & b \\ 0 & (d - x)(a - x) - cb \end{pmatrix},$$

which is noninvertible if and only if $(a - x) = 0$ or $(d - x)(a - x) - cb = 0$. But we assumed $x \neq a$ here. So in the case $x \neq a$ then $A - xI$ is noninvertible if and only if $x^2 - (a + d)x + (ad - bc) = 0$, as desired.
If, on the other hand, \( x = a \), then our original matrix would have been \( \begin{pmatrix} 0 & b \\ c & d - x \end{pmatrix} \), which is noninvertible (i.e. has rank \( \leq 1 \)) if and only if \( b = 0 \) or \( c = 0 \). But if \( x = a \), then \( x^2 - (a + d)x + (ad - bc) = bc \), so this is zero if and only if either \( b = 0 \) or \( c = 0 \).

So either way, we conclude that \( A - xI \) is noninvertible if and only if \( x^2 - (a + d)x + (ad - bc) = 0 \), i.e., if and only if \( x^2 - \text{tr} \ A \cdot x + \det \ A = 0 \).

(3) Using the result of the previous problem, i.e., that the eigenvalues of \( A \) are not dependent on the choice of basis of \( V \), we have

(a) If \( A \) has two distinct eigenvalues, then \( \text{tr}(A) \) is the sum of these two eigenvalues and \( \det(A) \) is the product of these two eigenvalues.

(b) Conclude that, if \( T \in \mathcal{L}(V) \) is a linear transformation, \( \dim V = 2 \), and \( T \) has two distinct eigenvalues, then for any choice of basis, \( \text{tr}(T) \) is the sum of these eigenvalues and \( \det(T) \) is the product of these eigenvalues. Thus the trace and determinant of \( T \) do not depend on the choice of basis.

Remark: We will see that this is more generally true (that \( \text{tr}(T) \) and \( \det(T) \) do not depend on the choice of basis of \( V \)) regardless of how many eigenvalues \( T \) has, and regardless of \( \dim V \) (when \( A \in \text{Mat}(n, n, F) \) for \( n \) not necessarily equal to 2, we define the trace of \( A \) as the sum of its diagonal entries, and we will define determinant by a more complicated formula).

A: (a) We saw that the eigenvalues are the roots of \( x^2 - \text{tr} \ A \cdot x + \det \ A \), so if there are two distinct eigenvalues, \( \lambda_1 \) and \( \lambda_2 \), then \( x^2 - \text{tr} \ A \cdot x + \det \ A = (x - \lambda_1)(x - \lambda_2) \). Then, \( \text{tr} \ A = \lambda_1 + \lambda_2 \) and \( \det \ A = \lambda_1 \lambda_2 \), as desired.

(b) If \( T \) is any linear transformation, then its eigenvalues are the same as those of \( \mathcal{M}(T) \) for every choice of basis. So the eigenvalues do not depend on the choice of basis. Since \( \text{tr} \mathcal{M}(T) \) is the sum of them and \( \det \mathcal{M}(T) \) is the product (at least in the case where there are two distinct eigenvalues), this implies that in the case there are two distinct eigenvalues, \( \text{tr} \mathcal{M}(T) \) and \( \det \mathcal{M}(T) \) do not depend on the choice of basis.

(4) Prove that the following are equivalent for \( T \in \mathcal{L}(V) \):

(a) \( V = \text{range}(T) + \text{null}(T) \).

(b) \( V = \text{range}(T) \oplus \text{null}(T) \).

(c) \( \text{range}(T) \cap \text{null}(T) = 0 \).

(d) Let \( S \in \mathcal{L}(\text{range}(T)) \) be defined by \( S(v) = T(v) \) (Basically, \( S = T|_{\text{range}(T)} : \text{range}(T) \to \text{range}(T) \)). Then, \( S \) is an isomorphism.

(e) In some basis of \( V \), \( \mathcal{M}(T) \) has a block matrix form

\[
\mathcal{M}(T) = \begin{pmatrix} A & 0_{m,n} \\ 0_{n,m} & 0_{n,n} \end{pmatrix},
\]

where \( A \in \text{Mat}(m, m, F) \) is invertible, and \( 0_{a,b} \in \text{Mat}(a, b, F) \) means the zero matrix of that size. Here we allow \( m = 0 \), in which case \( \mathcal{M}(T) \) is just the zero matrix.

A: Clearly, (b) implies (a) and (c). To show the reverse implications, note that the rank-nullity theorem implies that \( \dim \text{range}(T) + \dim \text{null}(T) = \dim V \). If (c) is true, then \( \text{range}(T) + \text{null}(T) = \text{range}(T) \oplus \text{null}(T) \), but by the above, this has dimension equal to \( \dim V \), so it must equal \( V \). If, instead, (a) is true, then we deduce immediately that \( V = \text{range}(T) \oplus \text{null}(T) \). Thus, (a), (b), and (c) are all equivalent.

Next, we show that (c) implies (d). Clearly, \( \text{null} \, S = \text{null} \, T \cap \text{range}(T) \). So (c) implies that \( S \) is injective, and hence it is an isomorphism (since \( \text{range}(T) \) is finite-dimensional). Conversely, if \( T|_{\text{range}(T)} \) is an isomorphism, then it is injective, so \( \text{null}(T) \cap \text{range}(T) = 0 \), and thus (d) implies (c). Therefore, (a)–(d) are all equivalent.
Let us show that (b) implies (e). Pick bases of range \(T\) and null(\(T\)). By (b), concatenating these (the basis for the range first, then the basis for the nullspace) yields a basis of \(V\). Let \(k = \text{rk}(T)\) and \(n = \dim V\). In this basis, the columns corresponding to the nullspace vectors, i.e., the last \(n - k\) columns, are all zero. The first \(k\) columns therefore span the column space, which has dimension \(k = \text{rk}(T)\). Since the column space is the span of the first \(k\) basis vectors, this means that the first \(k\) columns are zero below the upper-left \(k \times k\)-block. Call this block \(A\). Since the first \(k\) columns span the \(k\)-dimensional column space, they are linearly independent, and hence \(\text{rk}(A) = k\), i.e., \(A\) is invertible.

Conversely, we show (e) implies (b): assume \(k = \text{rk}(T) = \text{rk}(\mathcal{M}(T))\) and \(n = \dim V\), and \(\mathcal{M}(T)\) has the given form. Let the basis be \((v_1, \ldots, v_n)\). Clearly, \(v_{k+1}, \ldots, v_n \in \text{null} T\). Also, \(Tv_1, \ldots, Tv_k\) must span \(\text{range} T\) (since \(Tv_j = 0\) for \(j > k\)) and hence they are a basis for the range. Finally, since \(Tv_i \in \text{Span}(v_1, \ldots, v_k)\) for all \(i\), in fact \(\text{Span}(v_1, \ldots, v_k) = \text{range} T\). Hence, our basis of \(V\) is obtained by concatenating bases of range \(T\) and null \(T\), and we deduce (b). (We explained this in class; alternatively note that we immediately can deduce that \(\text{range}(T) \cap \text{null}(T) = \{0\}\), and hence (c), which we already explained is equivalent to (b)).

(5) Show that the linear transformation \(T \in \mathcal{L}(\text{Mat}(n, 1, F))\) given by left-multiplication by the following matrix does have the form of problem (4), and compute the block matrix form guaranteed by (4).(e):

\[
B := \begin{pmatrix}
1 & 2 & \cdots & n \\
1+1 & 2+1 & \cdots & 2n \\
\vdots & \vdots & \ddots & \vdots \\
(n-1)n+1 & (n-1)n+2 & \cdots & n^2 \\
\end{pmatrix}
\]

**Even if you couldn’t do Problem** (4), what you should so is: Show that in some basis of column vectors, the transformation \(v \mapsto Bv\) is written in the block form of (4).(e), and compute the block matrix that results. (In other words, in view of #17 from the last homework, \(B\) is conjugate to a matrix of the block form of (4).(e), which you compute).

**Hint:** Compute the null space and range separately and pick bases for these.

A: We will compute \(\mathcal{M}(S)\), where \(S \in \mathcal{L}(\text{range} T)\) is as defined in (d): basically, \(S = T|_{\text{range} T}\). If this is invertible, then (3).(d) is satisfied, and in view of (3).(b), we can extend our basis of range \(T\) to a basis of \(V\) by adjoining a basis of null \(T\); in this new basis the new matrix will be the block form of (3).(e), with the matrix we computed in the upper-left corner.

So first we compute the range. Let the columns of \(B\) be denoted \(w_1, \ldots, w_n\). Let \(v_1 := w_2 - w_1\), which is the all 1’s vector, and let \(v_2 := \frac{1}{n}(w_1 - v_1)\), which is the vector with entries \((0, 1, 2, \ldots, n - 1)\). Then, it is evident that

\[w_j = jv_1 + nv_2,\]

for all \(j\). So \((v_1, v_2)\) is a spanning list for range \(T\). Since \(v_1\) and \(v_2\) are not multiples of each other, it is also linearly independent and hence a basis.

Now we compute \(\mathcal{M}(v_1, v_2)(S)\). First we compute \(Sv_1\):

\[
Sv_1 = \begin{pmatrix}
1 + 2 + \cdots + n \\
n^2 + (1 + 2 + \cdots + n) \\
\vdots \\
(n-1)n^2 + (1 + 2 + \cdots + n)
\end{pmatrix} = \frac{n(n+1)}{2}v_1 + n^2v_2.
\]
Next we compute $Sv_2$. For this we will use the identity, where

$$\binom{m}{\ell} := \frac{m!}{\ell!(m-\ell)!} = \frac{m(m-1)\cdots(m-\ell+1)}{\ell(\ell-1)\cdots1},$$

with $\binom{m}{m} = \binom{m}{0} = 1$ by definition:

$$\binom{2}{2} + \binom{3}{2} + \cdots + \binom{k+1}{2} = \binom{k+2}{3}.$$

**Remark 0.3.** Note, more generally, $(\ell) + \cdots + \binom{k+\ell-1}{\ell} = \binom{k+\ell}{\ell}$. This identity has a combinatorial proof as follows: The RHS is the number of ways of picking an $\ell$-tuple $1 \leq a_1 < \cdots < a_\ell \leq k$ of increasing integers between 1 and $k$. Each term $\binom{\ell+\ell-1}{\ell}$ on the LHS is the number of ways of picking an $\ell$-tuple as above with the condition that $a_1 = j$. So the sum of all the terms on the LHS equals the total number of $\ell$-tuples as above, i.e., the RHS.

We proceed with the computation of $Sv_2$:

$$Sv_2 = \begin{pmatrix} 0 \cdot 1 + 1 \cdot 2 + \cdots + (n-1)n \\ n(0 + 1 + \cdots + (n-1)) + (0 \cdot 1 + 1 \cdot 2 + \cdots + (n-1)n) \\ 2n(0 + 1 + \cdots + (n-1)) + (0 \cdot 1 + 1 \cdot 2 + \cdots + (n-1)n) \\ \vdots \\ (n-1)n(0 + 1 + \cdots + (n-1)) + (0 \cdot 1 + 1 \cdot 2 + \cdots + (n-1)n) \end{pmatrix} = \begin{pmatrix} 2^{n+1} \\ n^2(n-1)/2 + 2^{n+1} \\ 2n^2(n-1)/2 + 2^{n+1} \\ \vdots \\ (n-1)n^2(n-1)/2 + 2^{n+1} \end{pmatrix} = 2\binom{n+1}{3}v_1 + \frac{n^2(n-1)}{2}v_2.$$

Hence, putting together the computations of $Sv_1$ and $Sv_2$, we obtain $A$ =

$$\mathcal{M}_{(v_1,v_2)}(S) = \begin{pmatrix} n(n+1)/2 & (n+1)n(n-1)/3 \\ n^2 & n^2(n-1)/2 \end{pmatrix}.$$
When \( n \geq 2 \), the eigenvalues of \( \mathcal{M}(T) \) and hence of \( B \) are the roots of the above polynomial (each corresponding to an eigenspace of dimension one), together with zero (in the case \( n > 2 \), occurring with an eigenspace of dimension \( n - 2 \)). The roots of the above polynomial are, by the quadratic equation,

\[
\frac{1}{2} \left( n(n^2 + 1)/2 \pm \sqrt{n^2(n^2 + 1)^2/4 - n^3(n + 1)(n - 1)/3} \right) = n(n^2 + 1)/4 \pm (n/12)\sqrt{3(n^2 - 2n + 3)(3n^2 + 2n + 1)}.
\]

Since the sum of the dimensions of the eigenspaces is \( \dim V \), it follows that \( V \) has an eigenbasis (see Proposition 5.21.(v)), obtained by concatenating bases for each of the eigenspaces.

(7) Axler, Ch. 6, p. 122, # 4

A: We apply the parallelogram identity:

\[ 4^2 + 6^2 = \|u + v\|^2 + \|u - v\|^2 = 2\langle u, u \rangle + 2\langle v, v \rangle = 2 \cdot 3^2 + 2\|v\|^2, \]

and hence \( \|v\|^2 = 34 \), so \( \|v\| = \sqrt{17} \).

(8) Axler, Ch. 6, p. 123, # 10

A: First, \( e_1 = 1 \) since this already has norm one: \( \int_0^1 1 \, dx = 1 \).

Next, \( e_2' = v_2 - \langle v_2, e_1 \rangle e_1 = x - \int x \, dx = x - \frac{1}{2} \). Then \( e_2 = e_2'/\|e_2\| \). Then

\[
\|e_2\|^2 = \int_0^1 (x - \frac{1}{2})^2 \, dx = (x^3/3 - x^2/2 + x/4)|_0^1 = 1/3 - 1/2 + 1/4 = 1/12.
\]

So \( e_2 = -\sqrt{3} + 2\sqrt{3} \cdot x \).

Finally,

\[
e_3' = v_3 - \langle v_3, e_1 \rangle e_1 - \langle v_3, e_2 \rangle e_2 = x^2 - \int_0^1 x^2 \, dx - (\sqrt{3} + 2\sqrt{3} \cdot x) \cdot \int_0^1 x^2(-\sqrt{3} + 2\sqrt{3} \cdot x) \, dx
\]

\[= x^2 - 1/3 - (\sqrt{3} + 2\sqrt{3} \cdot x) \cdot (\sqrt{3}/3 + \sqrt{3}/2)
\]

\[= x^2 - 1/3 + 1/2 - x = x^2 - x + 1/6.
\]

So \( e_3' = x^2 - x + 1/6 \). Then,

\[
\|e_3'\|^2 = \int_0^1 (x^2 - x + 1/6)^2 \, dx = \int_0^1 (x^4 - 2x^3 + \frac{4}{3}x^2 - \frac{1}{3}x + \frac{1}{36}) \, dx = 1/5 - 2/4 + 4/9 - 1/6 + 1/36 = 1/180.
\]

So \( e_3' = (6\sqrt{5}) \cdot (x^2 - x + 1/6) = \sqrt{5} - 6\sqrt{5}x + 6\sqrt{5} \cdot x^2 \).

(9) Next, consider the inner-product space \( V := \mathbb{C}\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}\} \) of complex-valued functions on the set \( \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}\} \). Equip \( V \) with the inner product

\[
\langle f, g \rangle := \frac{1}{n} \sum_{k=0}^{n-1} f\left(\frac{k}{n}\right) \overline{g\left(\frac{k}{n}\right)}.
\]

(a) Prove that the list \( (1, e^{2\pi i x}, \ldots, e^{2(n-1)\pi i x}) \) is an orthonormal basis of \( V \).

(b) Define the discrete Fourier coefficients of \( f \in V \) by

\[
\hat{f}_k := \frac{1}{n} \sum_{j=0}^{n-1} e^{-2\pi i (j/n)} f\left(\frac{j}{n}\right).
\]
Using Theorem 6.17, prove the discrete Fourier summation formula (now for all \( x \in \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}\} \)):

\[
f(x) = \sum_{k=0}^{n-1} \hat{f}_k e^{2k\pi i x}.
\]

A: (a) Note that

\[
\langle e^{2j\pi i x}, e^{2\ell\pi i x} \rangle = \frac{1}{n} \sum_{k=1}^{n-1} e^{2(j-\ell)\pi ik/n}.
\]

If \( n \mid (j-\ell) \), then the above sum is \( \frac{1}{n}(1 + \cdots + 1) = 1 \). If \( n \nmid (j-\ell) \), then the above sum is \( \frac{1}{n}(1 + \zeta + \zeta^2 + \cdots + \zeta^{n-1}) \) where \( \zeta^n = 1 \) but \( \zeta \neq 1 \). This is zero, since then \( 1 + \zeta + \cdots + \zeta^{n-1} = (1 - \zeta^n)(1 - \zeta)^{-1} = 0 \).

(b) By definition, \( \hat{f}_k = \langle f, e^{2k\pi i x} \rangle \). By (a) and (6.18), we deduce the statement.

(10) Axler, Ch. 6, p. 122, # 7, and deduce #6 from this.

A: We expand the numerator on the RHS:

\[
\|u + v\|^2 - \|u - v\|^2 + \|u + iv\|^2 i - \|u - iv\|^2 i
\]

\[
= \langle u + v, u + v \rangle - \langle u - v, u - v \rangle + \langle u + iv, u + iv \rangle i - \langle u - iv, u - iv \rangle i
\]

\[
= 2\langle u, v \rangle + 2\langle v, u \rangle + 2i\langle u, iv \rangle + 2i\langle iv, u \rangle
\]

\[
= 4 \text{ Re}(\langle u, v \rangle) + 4i \text{ Im}(\langle u, v \rangle).
\]

The last line is obtained by conjugate symmetry. Dividing by 4 yields the desired answer.

To deduce exercise 6, in the case \( F = \mathbb{R} \), dropping the final two terms of the above yields that \( \text{Re}(\langle u, v \rangle) = \langle u, v \rangle \) equals the RHS, which is the desired statement.