(1) Axler, Ch. 5, p. 94, #1

A: We need to show that, for all choices of \( u_i \in U_i \), \( T(u_1 + \cdots + u_m) \subseteq U_1 + \cdots + U_m \). Indeed, \( T(u_1 + \cdots + u_m) = T(u_1) + \cdots + T(u_m) \in U_1 + \cdots + U_m \).

(2) Axler, Ch. 5, p. 94, #4. Explain also: this statement is equivalent to: every eigenspace of \( T \) is \( S \)-invariant.

A: Suppose that \((T - \lambda I)v = 0\). We need to show that \((T - \lambda I)(Sv) = 0\). Indeed,

\[
(T - \lambda I)(Sv) = TSv - \lambda Sv = STv - \lambda Sv = S(T - \lambda I)v = S(0) = 0.
\]

For the final statement: note that the eigenspace \( \text{null}(T - \lambda I) \) of eigenvalue \( \lambda \) is \( S \)-invariant if and only if \( S(\text{null}(T - \lambda I)) \subseteq \text{null}(T - \lambda I) \), i.e., if \((T - \lambda I)v = 0\), then \( Sv \in \text{null}(T - \lambda I) \) as well.

(3) Axler, Ch. 5, p. 94, #7

A: First suppose that \( n = 1 \). Then, \( T = I \) and this evidently has only one eigenvalue, namely 1, with eigenbasis given by any nonzero vector.

Next, suppose \( n \geq 2 \). Let \( v_i \) be the standard basis vectors of \( \mathbb{F}^n \), i.e., which have 1 in the \( i \)-th component and zero elsewhere. Then, the \( T(v_i) \) are all equal. Hence, \( \text{range } T = \text{Span}(T(v_1), \ldots, T(v_n)) = \text{Span}(T(v_1)) \) is one-dimensional, and consists of the lists whose entries are all equal to each other. By the rank-nullity theorem, \( \dim \text{null } T = n - \text{rk}(T) = n - 1 \). So the nullspace has dimension \( n - 1 \). If we take any vector \( v \in \text{range } T \), it is easy to see that \( Tv = nv \). Hence, \( n \) is also an eigenvalue, and \( \text{range } T \) is a subspace of the \( n \)-eigenspace.

In particular, \( \text{range } T \cap \text{null } T = \{0\} \), so \( \text{range } T + \text{null } T = \text{range } T \oplus \text{null } T \). By the rank-nullity theorem, \( \dim \text{range } T + \dim \text{null } T = n \), and so \( V = \text{range } T \oplus \text{null } T \).

Since \( \text{null } T \) is the zero eigenspace and \( \text{range } T \) lies in the \( n \)-eigenspace, the following lemma implies that there are exactly two eigenvalues, 0 and \( n \), and their eigenspaces have dimensions \( n - 1 \) and 1, respectively:

**Lemma 0.1.** Suppose that \( V = U_1 \oplus \cdots \oplus U_m \) where each \( U_i \) consists of eigenvectors of \( V \) with eigenvalue \( \lambda_i \), and each \( U_i \neq 0 \). Further, assume the \( \lambda_i \) are all distinct. Then, the eigenvalues of \( T \) are exactly \( \lambda_1, \ldots, \lambda_m \), and their eigenspaces are exactly the \( U_i \).

**Proof.** Let \( W \subseteq V \) be the sum of all eigenspaces. We know that \( W = W_1 \oplus \cdots \oplus W_p \) where the \( W_i \) are the eigenspaces (Theorem 5.6, and the corollary from class). Hence, \( m \leq p \), and up to reindexing the \( W_i \), we can assume that the eigenvalue of \( W_i \) is \( \lambda_i \) when \( i \leq m \). Thus, \( W_i \supseteq U_i \), so \( \dim W_i \geq \dim U_i \).

Then,

\[
n \geq \dim W = \dim W_1 + \cdots + \dim W_p \geq \dim U_1 + \cdots + \dim U_m = \dim V = n.
\]

Hence, \( W_i = U_i \) for all \( i \leq m \), and \( p = m \), as desired. \( \square \)

(4) (a) Axler, Ch. 5, p. 94, #8

(b) Similarly, find all eigenvalues and eigenvectors of the *forward shift operator*, \( T(z_1, z_2, z_3, \ldots) = (0, z_1, z_2, \ldots) \).
Similarly, prove that the following are equivalent:

Prove the following generalization of a result in Tuesday’s class:

Axler, Ch. 5, p. 95, #17

A: Let \((v_1, \ldots, v_n)\) be a basis for \(V\) such that \(\mathcal{M}(V)\) is upper-triangular in this basis. Then, \(Tv_j \in \text{Span}(v_1, \ldots, v_j)\) for all \(j\) (this follows immediately from the form of the matrix, or refer to Proposition 5.12.(b)). Hence, \(T(\text{Span}(v_1, \ldots, v_j)) \subseteq \text{Span}(v_1, \ldots, v_j)\), so that \(\text{Span}(v_1, \ldots, v_j)\) is invariant. It is also \(j\)-dimensional, since the list \((v_1, \ldots, v_j)\) is linearly independent, being a sublist of the linearly independent list \((v_1, \ldots, v_n)\).

Prove the following generalization of a result in Tuesday’s class: \(T\) has an eigenbasis with at most one nonzero eigenvalue if and only if \(T = \lambda P_{U,W}\) for some \(\lambda \in \mathbb{F}\) and some \(U, W \subseteq V\).

(Recall the definition of eigenbases and projection operators \(P_{U,W}\) from Axler, Ch. 5: an eigenbasis is a basis \((v_1, \ldots, v_n)\) of \(V\) such that each \(v_i\) is an eigenvector of \(V\); given \(V = U \oplus W\), then \(P_{U,W} \in \mathcal{L}(V)\) is the operator given by \(P_{U,W}(u + w) = u\) for all \(u \in U\) and \(w \in W\).)

A: If \(T\) has an eigenbasis with at most one nonzero eigenvalue, let \(U\) be the span of the eigenbasis vectors of nonzero eigenvalue, and let \(W\) be the span of the eigenbasis vectors of zero eigenvalue. Then since bases for \(U\) and \(W\) combine to form a basis of \(V\), \(V = U \oplus W\) (indeed, \(\dim V = \dim U + \dim W\) and \(V = U + W\); recall Proposition 2.19). If zero is the only eigenvalue of vectors in the eigenbasis, then \(V = W\) and \(T = 0\), so \(T = P_{\{0\},V} = P_{U,W}\), which is of the desired form. If not, let \(\lambda \neq 0\) be the nonzero eigenvalue that occurs. We claim that \(T = \lambda P_{U,W}\). Indeed, for \(u \in U\), we can write \(u\) as a sum of eigenbasis vectors of eigenvalue \(\lambda\), so \(Tu = \lambda u\), whereas for \(w \in W\), similarly we have \(Tw = 0\). Hence \(T(u + w) = \lambda = \lambda P_{U,W}u\) for all \(u \in U\) and \(w \in W\). Therefore \(T = \lambda P_{U,W}\).

Similarly, prove the following are equivalent:

(a) An operator \(T\) is of the form \(T = P_{U,W}\) for some \(U, W\);
(b) \(T = P_{\text{range } T, \text{null } T}\);
(c) \(T^2 = T\).

Bonus: Generalize this statement to the case of multiples of projection operators, i.e., \(\lambda P_{U,W}\). This should be in terms of \(\lambda\); conditions (a), (b), and (c) need to be suitably modified.

A: We first show that (a) implies (b): if \(T = P_{U,W}\), then \(P_{U,W}(u + w) = 0\) if and only if \(u = 0\), i.e., if and only if \(u + w \in W\). Hence \(\text{null}(P_{U,W}) = W\). Also, \(\text{range } P_{U,W} = U\), by definition. So \(T = P_{\text{range } T, \text{null } T}\).

On the other hand, it is obvious that (b) implies (a), for \(U = \text{range } T\) and \(W = \text{null } T\). So (a) and (b) are equivalent.

Next we prove that (a) implies (c). \(P_{U,W}^2(u + w) = P_{U,W}(u) = u\). So \(P_{U,W}^2 = P_{U,W}\).

Finally we show that (a) implies (b). If \(T^2 = T\), then for \(u \in \text{range } T\), say \(u = T(v)\) for some \(v\); then \(T(u) = T^2(v) = T(v) = u\). Next, for \(w \in \text{null } T\), clearly \(T(w) = 0\). So if we can show that \(V = \text{range } T \oplus \text{null } T\), we would be done. Well, first note that \(\text{range } T \cap \text{null } T = 0\), since for all \(u \in \text{range } T\), \(T(u) = u\), which is nonzero if \(u\) is nonzero.
(Alternatively, range $T$ lies in the eigenspace of eigenvalue one, and so it cannot intersect the null space, which is the eigenspace of eigenvalue zero). Next, we need to show that $V = \text{range } T + \text{null } T$. For this note that, for all $v \in V$, $v = T(v) + (v - T(v))$. So if we can show that $v - T(v) \in \text{null } T$, we would be done. However, $T(v - T(v)) = T(v) - T^2(v) = 0$, so this proves the needed assertion.

(8) In the next three exercises, using the complex conjugation map, we will prove a result stated in class and in Axler in a different way, using complex conjugation:

**Theorem 0.2.** If $F = \mathbb{R}$, then any linear transformation $T \in \mathcal{L}(V)$, for $V$ finite-dimensional and nonzero, admits an invariant subspace $U \subseteq V$ such that $\dim U \in \{1,2\}$.

First, take a basis $(v_1, \ldots, v_n)$ of $V$. By replacing $T$ with $\mathcal{M}(T)$, show that the theorem is equivalent to the following one:

**Theorem 0.3.** Let $A \in \text{Mat}(n,n,\mathbb{R})$ for $n \geq 1$. Then there exists a nonzero subspace $U \subseteq \text{Mat}(n,1,\mathbb{R})$ with $\dim U \leq 2$, such that for all $u \in U$, $Au \in U$ (i.e., $U$ is invariant under $T_A : \text{Mat}(n,1,\mathbb{R}) \to \text{Mat}(n,1,\mathbb{R})$, $T_A(u) = Au$).

A: Assume the first theorem. Let $A \in \text{Mat}(n,n,\mathbb{R})$ for $n \geq 1$. Set $T := T_A$. Then $\mathcal{M}(T) = A$. Let $U \subseteq \text{Mat}(n,1,\mathbb{R})$ be as given in the first theorem. Then, for all $u \in U$, $T(u) \in U$ implies that $Au \in U$. So the second theorem holds.

Similarly, assume the second theorem. Take $T \in \mathcal{L}(V)$ for any nonzero finite-dimensional vector space $V$, and pick any basis of $V$. Let $A := \mathcal{M}(A)$. Let $U \subseteq \text{Mat}(n,1,\mathbb{R})$ be a 1 or 2-dimensional subspace of $\text{Mat}(n,1,\mathbb{R})$. Then $\mathcal{M}^{-1}(U) \subseteq V$ is a 1 or 2-dimensional subspace of $V$, and it is $T$-invariant since, for all $u \in U$, $\mathcal{M}(TM^{-1}(u)) = \mathcal{M}(T)u = Au \in U$. So $TM^{-1}(u) \in \mathcal{M}^{-1}(U)$. Thus $\mathcal{M}^{-1}(U)$ is $T$-invariant.

Now, take a matrix $A \in \text{Mat}(n,n,\mathbb{R})$ ($n \geq 1$) and consider it as a complex matrix.

(a) Explain why there is a nonzero eigenvector $v \in \text{Mat}(n,1,\mathbb{C})$, $Av = \lambda v$ for some $\lambda \in \mathbb{C}$.

(b) If $\lambda \in \mathbb{R}$, prove that there actually exists a nonzero $v \in \text{Mat}(n,1,\mathbb{R})$ such that $Av = \lambda v$, i.e., $A$ has a nonzero real eigenvector. (Hint: Explain that $A - \lambda I$ is noninvertible, and conclude that, since $A - \lambda I \in \text{Mat}(n,n,\mathbb{R})$, working over $F = \mathbb{R}$, one still has $\text{null}(A - \lambda I) \neq \{0\}$.)

A: a) This is a consequence of Theorem 5.10, since $V$ is nonzero and finite-dimensional. More precisely, that would give a nonzero eigenvector of $T_A$, which is then an eigenvector of $A$.

b) If $\lambda \in \mathbb{R}$, since $A - \lambda I$ is not invertible by a complex matrix, it is certainly not invertible by a real matrix either. Hence, $A - \lambda I$ has rank less than $n$ even working with $F = \mathbb{R}$. And hence $\text{null}(A - \lambda I) \neq \{0\}$, again as a real matrix. So there is a nonzero column vector with real entries in the null space of $A - \lambda I$, i.e., a nonzero real eigenvector.

(10) Finally, we handle the case that $\lambda \notin \mathbb{R}$. For this, recall complex conjugation: if $z = a + bi \in \mathbb{C}$, then we define $\bar{z} := a - bi$ (here $i$ is the imaginary number $i = \sqrt{-1}$). Recall that this satisfies $\overline{\overline{z}} = z \cdot \overline{w}$ and $\overline{z} + \overline{w} = \bar{z} + \bar{w}$.

We extend this to matrices by taking conjugates of all entries: if $B = (b_{ij}) \in \text{Mat}(m,n,\mathbb{C})$, then $\overline{B} := (\overline{b_{ij}})$. Then also $\overline{BC} = \overline{B}C$ and $\overline{B + C} = \overline{B} + \overline{C}$ (whenever these make sense).

(a) Now suppose that $Av = \lambda v$ for $A \in \text{Mat}(n,n,\mathbb{R})$, for nonzero $v \in \text{Mat}(n,1,\mathbb{C})$, and $\lambda \in \mathbb{C}$ which is not real. Show that $A\overline{v} = \overline{\lambda v}$, so that $\overline{v} \in \text{Mat}(n,1,\mathbb{C})$ is also a nonzero eigenvector of $A$.

(b) In the situation above, conclude that $v$ and $\overline{v}$ are linearly independent (over $\mathbb{C}$), since $\lambda \neq \overline{\lambda}$. Thus, conclude that $\text{Span}_\mathbb{C}(v, \overline{v})$ is two-dimensional. Here $\text{Span}_\mathbb{C}$ is to remind you that we are working over $\mathbb{C}$ (inside the space $\text{Mat}(n,1,\mathbb{C})$).

(c) Show that $\text{Span}_\mathbb{C}(v, \overline{v}) = \text{Span}_\mathbb{C}(v + \overline{v}, i(v - \overline{v}))$ (where $i$ is the imaginary number).
(d) Next, prove that $v + \bar{v} \in \text{Mat}(n,1,\mathbb{R})$ and $i(v - \bar{v}) \in \text{Mat}(n,1,\mathbb{R})$.

(e) Conclude from the above that, working over $\mathbb{R}$, $U := \text{Span}_\mathbb{R}(v + \bar{v}, i(v - \bar{v}))$ is a a two-dimensional real subspace of $\text{Mat}(n,1,\mathbb{R})$ (since the two vectors are linearly independent over $\mathbb{C}$ and hence also over $\mathbb{R}$), which is invariant under $T_A$. This completes the proof of the theorem.

A: (a) Since complex conjugation is multiplicative, $A\bar{v} = \bar{\lambda v} = \bar{\lambda} \bar{v}$, but also, $A\bar{v} = \bar{A\bar{v}} = \bar{A} \bar{v}$ ($A = A$ since $A$ is real). Hence $A\bar{v} = \bar{A} \bar{v}$, as desired.

(b) Since $\lambda, \bar{\lambda}$ and $v$ is by definition nonzero, so is $\bar{v}$, and Theorem 5.10 implies that $(v, \bar{v})$ is linearly independent (in $\text{Mat}(n,1,\mathbb{C})$, over $F = \mathbb{C}$). So $\text{Span}_\mathbb{C}(v, \bar{v})$ is 2-dimensional.

(c) Clearly $\text{Span}_\mathbb{C}(v + \bar{v}, i(v - \bar{v})) \subseteq \text{Span}_\mathbb{C}(v, \bar{v})$. For the opposite inclusion, note that $v = \frac{1}{2}(v + \bar{v}) - \frac{1}{2}(i(v - \bar{v}))$ and $\bar{v} = \frac{1}{2}(v + \bar{v}) + \frac{i}{2}(i(v - \bar{v}))$. So the two spans are equal.

(d) Note that $w$ has real coefficients if and only if $\bar{w} = w$. So we note that $\bar{v} + v = \bar{v} + \bar{v} = \bar{v} + v = v + \bar{v}$. Similarly, $i\bar{(v - \bar{v})} = -i(v - v)$ (since complex conjugation is multiplicative), and this is $-i(\bar{v} - v) = -i(v - \bar{v})$. So both of the vectors mentioned have real coefficients, i.e., $\bar{v} + v, i(v - \bar{v}) \in \text{Mat}(n,1,\mathbb{R})$.

(e) Write $u := v + \bar{v}$ and $w := i(v - \bar{v})$. Since $(u, w)$ is linearly independent over $\mathbb{C}$, it is also linearly independent over $\mathbb{R} \subset \mathbb{C}$, by definition of linear independence. So $\text{Span}_\mathbb{R}(u, w)$ is a two-dimensional real subspace. Next we claim that $\text{Span}_\mathbb{R}(u, w)$ is $A$-invariant. Clearly $A \text{Span}_\mathbb{R}(u, w) \subseteq \text{Span}_\mathbb{C}(u, w) = \text{Span}_\mathbb{C}(v, \bar{v})$, since the latter is $A$-invariant and $\text{Span}_\mathbb{R}(u, w) \subseteq \text{Span}_\mathbb{C}(u, w) = \text{Span}_\mathbb{C}(v, \bar{v})$. On the other hand, since $A, u$, and $w$ have real entries, $A \text{Span}_\mathbb{R}(u, w) \subseteq \text{Mat}(n,1,\mathbb{R})$. To conclude, we claim that $\text{Mat}(n,1,\mathbb{R}) \cap \text{Span}_\mathbb{C}(u, w) = \text{Span}_\mathbb{R}(u, w)$. Indeed, for this it is enough to show that, if $(a + bi)u + (c + di)w \in \text{Mat}(n,1,\mathbb{R})$ for some $a, b, c, d \in \mathbb{R}$, then $c = 0 = d$. We have $(a + bi)u + (c + di)w = (au + cw) + i(bu + dw)$. For this to have real entries, that means that $bu + dw = 0$. But since $u$ and $w$ are linearly independent, that implies $b = d = 0$, as desired. Hence, $\text{Mat}(n,1,\mathbb{R}) \cap \text{Span}_\mathbb{C}(u, w) = \text{Span}_\mathbb{R}(u, w)$, and since $A \text{Span}_\mathbb{R}(u, w)$ is in this, that implies that $A \text{Span}_\mathbb{R}(u, w) \subseteq \text{Span}_\mathbb{R}(u, w)$, and $\text{Span}_\mathbb{R}(u, w)$ is $A$-invariant.