The first half of this problem set is largely a reading assignment (the Gaussian elimination handout) plus some computations. The reason for this is that we won’t have class on Tuesday, and the week after this is the midterm, so this seems to be the best way for you to get prepared for the test. We will try to cover as much of this material as possible on Thursday (the day before the problem set is due).

(1) Perform Gaussian elimination on the following matrix. Your answer should include all of the steps as well as the final row echelon matrix.

\[
\begin{pmatrix}
0 & 2 & 4 & -5 \\
3 & 0 & 6 & 12 \\
-2 & 5 & 10 & 8
\end{pmatrix}
\]

A: Steps:

(a) Swap the first and second rows:

\[
\begin{pmatrix}
3 & 0 & 6 & 12 \\
0 & 2 & 4 & -5 \\
-2 & 5 & 10 & 8
\end{pmatrix}
\]

(b) Divide the first row by 3:

\[
\begin{pmatrix}
1 & 0 & 2 & 4 \\
0 & 2 & 4 & -5 \\
-2 & 5 & 10 & 8
\end{pmatrix}
\]

(c) Add twice the first row to the third:

\[
\begin{pmatrix}
1 & 0 & 2 & 4 \\
0 & 2 & 4 & -5 \\
0 & 5 & 14 & 16
\end{pmatrix}
\]

(d) Divide the second row by 2:

\[
\begin{pmatrix}
1 & 0 & 2 & 4 \\
0 & 1 & 2 & -5/2 \\
0 & 5 & 14 & 16
\end{pmatrix}
\]

(e) Subtract 5 times the second row from the third:

\[
\begin{pmatrix}
1 & 0 & 2 & 4 \\
0 & 1 & 2 & -5/2 \\
0 & 0 & 4 & 57/2
\end{pmatrix}
\]

(f) Divide the third row by 4:

\[
\begin{pmatrix}
1 & 0 & 2 & 4 \\
0 & 1 & 2 & -5/2 \\
0 & 0 & 1 & 57/8
\end{pmatrix}
\]
(2) Suppose we are given an equation of the form \( Ax = b \) where \( A \in \text{Mat}(m, n, \mathbb{F}) \) is an \( m \times n \) matrix (reminder: this is what \( A \in \text{Mat}(m, n, \mathbb{F}) \) means), \( b \in \text{Mat}(m, 1, \mathbb{F}) \) is a column vector, and 

\[
x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}
\]

is a column vector of indeterminants (i.e., variables) \( x_1, \ldots, x_n \). Writing

\[
A = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix},
\]

the equation is the same as the system of equations

\[
a_{1,1}x_1 + \cdots + a_{1,n}x_n = b_1 \\
\vdots \\
a_{m,1}x_1 + \cdots + a_{m,n}x_n = b_m.
\]

Form the \textit{augmented} matrix

\[
(A|b) = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m,1} & \cdots & a_{m,n} & b_m \end{pmatrix}.
\]

Here augmented means that it is a matrix with a line separating most of the matrix from the final column. We think of the final column (to the right of the line) a bit differently from the rest of the matrix.

Show that, if we perform an elementary row operation (see the lecture notes or the Gaussian elimination handout) to this augmented matrix, the set of solutions to the corresponding equation \( Ax = b \) does not change. Refer to the Gaussian elimination handout (particularly the section on solving systems of equations).

A: This is mostly a matter of understanding the definitions. Namely, if \( Ax = b \), then a row operation is of the form \((A|b) \mapsto (EA|Eb)\) for some elementary matrix \( E \), which is invertible since the inverse row operation produces the inverse matrix \( E^{-1} \). Now, if \( Ax = b \), then also \( EAx = Eb \), so a row operation takes a solution of \( Ax = b \) to a solution of \( EAx = Eb \). On the other hand, if \( EAx = Eb \), then \( E^{-1}EAx = E^{-1}Eb \), and hence \( Ax = b \). Thus \( Ax = b \) if and only if \((EA)x = Eb\). Therefore row operations \((A|b) \mapsto (EA|Eb)\) do not change the space of solutions.

(3) Form the augmented matrix for the following system, and perform Gaussian elimination on it. Then, using your final augmented matrix in row echelon form, express the set of solutions to the equations as explained in the Gaussian elimination handout:

\[
\begin{align*}
x_1 + 3x_2 - 5x_3 + 2x_4 &= 10 \\
x_1 + 4x_2 + x_3 + x_4 &= 7 \\
-2x_1 + 3x_2 - 5x_3 - 2x_4 &= 3.
\end{align*}
\]
A:
\[
\begin{pmatrix}
1 & 3 & -5 & 2 & 10 \\
1 & 4 & 1 & 1 & 7 \\
-2 & 3 & -5 & -2 & 3
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 3 & -5 & 2 & 10 \\
0 & 1 & 6 & -1 & -3 \\
-2 & 3 & -5 & -2 & 3
\end{pmatrix}
\rightarrow
\begin{pmatrix}
0 & 1 & 6 & -1 & -3 \\
0 & 9 & -15 & 2 & 23 \\
1 & 3 & -5 & 2 & 10
\end{pmatrix}
\rightarrow
\begin{pmatrix}
0 & 1 & 6 & -1 & -3 \\
0 & 9 & -15 & 2 & 23 \\
0 & 0 & -69 & 11 & 50
\end{pmatrix}
\rightarrow
\begin{pmatrix}
0 & 0 & 1 & -11/69 & -50/69 \\
0 & 9 & -15 & 2 & 23 \\
1 & 3 & -5 & 2 & 10
\end{pmatrix}
\rightarrow
\begin{pmatrix}
0 & 0 & 1 & -11/69 & -50/69 \\
0 & 9 & -15 & 2 & 23 \\
0 & 0 & -69 & 11 & 50
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 3 & -5 & 2 & 10 \\
0 & 1 & 6 & -1 & -3 \\
0 & 0 & -69 & 11 & 50
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 3 & -5 & 2 & 10 \\
0 & 1 & 6 & -1 & -3 \\
0 & 0 & -69 & 11 & 50
\end{pmatrix}.
\]

So there is one free variable, \(x_4\). The solution set for \((x_1, x_2, x_3, x_4)\) is:
\[
\{(10 - 2x_4 + 5((11/69)x_4 - 50/69) - 3(-3 + x_4 - 6((11/69)x_4 - 50/69)), \\
- 3 + x_4 - 6((11/69)x_4 - 50/69), (11/69)x_4 - 50/69, x_4) : x_4 \in \mathbb{F}\}
\]
\[
= \{(-4/3)x + 7/3, (1/23)x + 31/23, (11/69)x_4 - 50/69, x_4) : x_4 \in \mathbb{F}\}.
\]
(Here we are assuming that \(\mathbb{F} = \mathbb{R}\) or \(\mathbb{C}\); it would be valid also for many other fields, but not all fields. More precisely the answer above is valid for any field in which 69 is nonzero.)

(4) Read about Gauss-Jordan elimination in the Gaussian elimination handout (this is an additional step to Gaussian elimination which allows you to make more entries of your matrix zero at the end).

Consider the matrix (with two matrices put together)
\[
\begin{pmatrix}
2 & 6 & 5 & 1 & 0 & 0 \\
1 & 3 & 2 & 0 & 1 & 0 \\
2 & 5 & 1 & 0 & 0 & 1
\end{pmatrix}
\]

(a) Explain why, if you perform this on the matrix on the left of the vertical line while also doing the same operations to the matrix on the right of the vertical line, the end result is the identity matrix on the left of the line and the inverse of the original matrix on the right.

(b) Do it in this example, going from \((A|I)\) as above (where \(I\) is the identity matrix) to \((I|B)\), by Gauss-Jordan elimination on the \(A\) part (doing the same row operations to the matrix on the right of the vertical line simultaneously).

(c) Verify that the end result, \((I|B)\), indeed satisfies \(AB = BA = I\).

A: (a) This is because, if we perform Gauss-Jordan elimination to \(A\), we obtain a sequence of operations corresponding to elementary matrices \(E_1, E_2, \ldots, E_m\), such that \(E_mE_{m-1} \cdots E_1A = C\) has reduced row echelon form. It also has the same rank as \(A\). If \(A\) is invertible, then \(C = I\). In this case, \(E_mE_{m-1} \cdots E_1A^{-1}\). Thus if we perform the same operations to \(I\), we get \(E_mE_{m-1} \cdots E_1I = A^{-1}\).

Hence, we just have to verify that \(A\) is invertible, i.e., that the obtained reduced row echelon matrix \(C\) is the identity. We will do this in (b).

(b) Here are the steps:

(a) Divide the first row by two:
\[
\begin{pmatrix}
1 & 3 & 5/2 & 1/2 & 0 & 0 \\
1 & 3 & 2 & 0 & 1 & 0 \\
2 & 5 & 1 & 0 & 0 & 1
\end{pmatrix}
\]

(b) Subtract the first row from the second:
\[
\begin{pmatrix}
1 & 3 & 5/2 & 1/2 & 0 & 0 \\
0 & 0 & -1/2 & -1/2 & 1 & 0 \\
2 & 5 & 1 & 0 & 0 & 1
\end{pmatrix}
\]
(c) Subtract twice the first row from the third:
\[
\begin{pmatrix}
1 & 3 & 5/2 \\
0 & 0 & -1/2 \\
0 & -1 & -4
\end{pmatrix}
\begin{pmatrix}
1/2 & 0 & 0 \\
-1/2 & 1 & 0 \\
-1 & 0 & 1
\end{pmatrix}
\]

(d) Swap the second and third rows:
\[
\begin{pmatrix}
1 & 3 & 5/2 \\
0 & -1 & -4 \\
0 & 0 & -1/2
\end{pmatrix}
\begin{pmatrix}
1/2 & 0 & 0 \\
-1 & 0 & 1 \\
-1/2 & 1 & 0
\end{pmatrix}
\]

(e) Negate the second row:
\[
\begin{pmatrix}
1 & 3 & 5/2 \\
0 & 1 & 4 \\
0 & 0 & -1/2
\end{pmatrix}
\begin{pmatrix}
1/2 & 0 & 0 \\
1 & 0 & -1 \\
-1/2 & 1 & 0
\end{pmatrix}
\]

(f) Multiply the third row by \(-2\):
\[
\begin{pmatrix}
1 & 3 & 5/2 \\
0 & 1 & 4 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1/2 & 0 & 0 \\
1 & 0 & -1 \\
1 & -2 & 0
\end{pmatrix}
\]

(g) Subtract 4 times the third row from the second:
\[
\begin{pmatrix}
1 & 3 & 5/2 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1/2 & 0 & 0 \\
-3 & 8 & -1 \\
1 & -2 & 0
\end{pmatrix}
\]

(h) Subtract 5/2 times the third row from the first:
\[
\begin{pmatrix}
1 & 3 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
-2 & 5 & 0 \\
-3 & 8 & -1 \\
1 & -2 & 0
\end{pmatrix}
\]

(i) Subtract 3 times the second row from the first:
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
7 & -19 & 3 \\
-3 & 8 & -1 \\
1 & -2 & 0
\end{pmatrix}
\]

(c) Indeed,
\[
\begin{pmatrix}
2 & 6 & 5 \\
1 & 3 & 2 \\
2 & 5 & 1
\end{pmatrix}
\begin{pmatrix}
7 & -19 & 3 \\
-3 & 8 & -1 \\
1 & -2 & 0
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
7 & -19 & 3 \\
-3 & 8 & -1 \\
1 & -2 & 0
\end{pmatrix}
\begin{pmatrix}
2 & 6 & 5 \\
1 & 3 & 2 \\
2 & 5 & 1
\end{pmatrix}
\]

(5) Using the information in the Gaussian elimination handout, perform PLB decomposition on the following matrix \(A\) (here, \(B\) should be the row echelon matrix you obtain using Gaussian elimination: this is not Gauss-Jordan elimination and so the matrix should \textit{not} necessarily be in reduced row echelon form):

\[
\begin{pmatrix}
0 & 0 & 2 & 1 \\
1 & 2 & -2 & 4 \\
2 & -1 & -5 & 3
\end{pmatrix}
\]

\(A\): We first have to perform Gaussian elimination to figure out what the permutations are:
(a) Swap the first and second row:
\[
\begin{pmatrix}
1 & 2 & -2 & 4 \\
0 & 0 & 2 & 1 \\
2 & -1 & -5 & 3
\end{pmatrix}
\]
(b) Subtract twice the first row from the third:
\[
\begin{pmatrix}
1 & 2 & -2 & 4 \\
0 & 0 & 2 & 1 \\
0 & -5 & -1 & -5
\end{pmatrix}
\]
(c) Swap the second and third rows:
\[
\begin{pmatrix}
1 & 2 & -2 & 4 \\
0 & -5 & -1 & -5 \\
0 & 0 & 2 & 1
\end{pmatrix}
\]
(d) Divide the second row by $-5$:
\[
\begin{pmatrix}
1 & 2 & -2 & 4 \\
0 & 1 & 1/5 & 1 \\
0 & 0 & 2 & 1
\end{pmatrix}
\]
(e) Divide the third row by 2:
\[
\begin{pmatrix}
1 & 2 & -2 & 4 \\
0 & 1 & 1/5 & 1 \\
0 & 0 & 1 & 1/2
\end{pmatrix}
\]

Now, we can do the same thing to the matrix we get by first swapping the first and second rows, and then the second and third:
\[
\begin{pmatrix}
1 & 2 & -2 & 4 \\
2 & -1 & -5 & 3 \\
0 & 0 & 2 & 1
\end{pmatrix}
\]
\[
\begin{pmatrix}
1 & 2 & -2 & 4 \\
0 & -5 & -1 & -5 \\
0 & 0 & 2 & 1
\end{pmatrix}
\]
\[
\begin{pmatrix}
1 & 2 & -2 & 4 \\
0 & 1 & 1/5 & 1 \\
0 & 0 & 2 & 1
\end{pmatrix}
\]
\[
\begin{pmatrix}
1 & 2 & -2 & 4 \\
0 & 1 & 1/5 & 1 \\
0 & 0 & 1 & 1/2
\end{pmatrix}
\]

The permutation matrix we multiplied by was $P^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$. The row operations we performed was multiplication by the matrix (we can get this by applying the same operations to the identity matrix): $L^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2/5 & -1/5 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}$. Then $L^{-1}P^{-1}A = B = \begin{pmatrix} 1 & 2 & -2 & 4 \\ 0 & 1 & 1/5 & 1 \\ 0 & 0 & 1 & 1/2 \end{pmatrix}$. Now we just need to invert $P^{-1}$ and $L^{-1}$. We can do this either with Gauss-Jordan elimination, or by performing the opposite row operations we used to get them, in the opposite order, to the identity matrix (these are of course the same thing):
\[
P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -5 & 0 \\ 0 & 0 & 2 \end{pmatrix} \]
Hence, $A = PLB$ is
\[
\begin{pmatrix}
0 & 0 & 2 & 1 \\
1 & 2 & -2 & 4 \\
2 & -1 & -5 & 3
\end{pmatrix}
= \begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
2 & -5 & 0 \\
0 & 0 & 2
\end{pmatrix}
\begin{pmatrix}
1 & 2 & -2 & 4 \\
1 & 1/5 & 1 \\
0 & 0 & 1/2
\end{pmatrix}.
\]

(6) Recall Google PageRank from Lecture 1 (see the notes under Schedule of the website if necessary). Find all possible (positive!) PageRank rankings of the websites $a, b, c, d$ according to the following diagram. To do this, use Gaussian elimination: to solve the equation $Av = v$, it is the same as finding the null space of $(A - I)$. You should explain why this is, and compute the null space of the appropriate matrix using Gaussian elimination, showing your work.

Hint: Refer to the Gaussian elimination handout on computing the null space (we explained briefly how to do this in class (refer to the lecture notes): the null space is the same as the null space of the row echelon matrix you obtain by applying Gaussian elimination, and this is expressed by arbitrarily picking the free entries of the column vectors and solving uniquely for the pivot entries so that the resulting column vector is in the null space.)

![Diagram](attachment:diagram.png)

A: We need to solve the equation $Av = v$, where $A$ is the matrix
\[
A = \begin{pmatrix}
0 & 1/2 & 1/3 & 0 \\
1/2 & 0 & 1/3 & 1/2 \\
0 & 0 & 0 & 1/2 \\
1/2 & 1/2 & 1/3 & 0
\end{pmatrix}.
\]

Note that $Av = v$ if and only if $(A - I)v = 0$, so we need to compute the null space of $(A - I)$. Now, we compute this null space by Gaussian elimination on $A - I$:
\[
\begin{pmatrix}
-1 & 1/2 & 1/3 & 0 \\
1/2 & -1 & 1/3 & 1/2 \\
0 & 0 & -1 & 1/2 \\
1/2 & 1/2 & 1/3 & -1
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
1 & -1/2 & -1/3 & 0 \\
1/2 & -1 & 1/3 & 1/2 \\
0 & 0 & -1 & 1/2 \\
1/2 & 1/2 & 1/3 & -1
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
1 & -1/2 & -1/3 & 0 \\
0 & -3/4 & 1/2 & 1/2 \\
0 & 0 & -1 & 1/2 \\
0 & 3/4 & 1/2 & -1
\end{pmatrix}.
\]

We deduce that the null space is
\[
\left\{ \begin{pmatrix}
\frac{2}{3}x \\
x \\
\frac{1}{2}x \\
x
\end{pmatrix} : x \in \mathbb{R} \right\} = \left\{ \begin{pmatrix}
4y \\
6y \\
3y \\
6y
\end{pmatrix} : y \in \mathbb{R} \right\}.
\]

So for $y \geq 1$, these are all the possible rank functions. Note that $b$ and $d$ get the highest ranking, followed by $a$, then $c$.

(7) Suppose that $T \in \mathcal{L}(V)$ is a linear transformation satisfying $T^m = 0$ for some $m \geq 1$. Show that $T$ is not injective.
A: Let us assume that $m$ is minimal for this property. Hence, $T^{m-1} \neq 0$ (where, by definition, $T^0 = I$, which is nonzero since $V \neq 0$). Then, for some $v \in V$, $T^{m-1}v \neq 0$. But, $T(T^{m-1}v) = 0$. Hence, $T \neq 0$, so $T$ is not injective.

(8) Next, suppose that $T^2 = 0$. Show that $\text{rk}(T) \leq \frac{\dim V}{2}$. (Bonus: Generalize this inequality to the case $T^m = 0$: show that $\text{rk}(T) \leq \frac{(m-1)\dim V}{m}$.)

A: Note first that $T^2 = 0$ is equivalent to

$$T|_{\text{range } T} = 0.$$  

The rank-nullity theorem applied to $T$ yields

$$\dim V = \dim \text{null}(T) + \text{rk} T.$$  

By the first equation, $\text{range } T = \text{null}(T|_{\text{range } T}) \subseteq \text{null}(T)$. Hence $\dim \text{null } T \geq \dim \text{range } T = \text{rk} T$. We conclude that

$$\dim V \geq \text{rk } T + \text{rk } T = 2 \text{rk } T,$$

and hence $\text{rk } T \leq \frac{\dim V}{2}$, as desired.

Bonus: We can apply induction. Since $T^m = 0$, we conclude that $T^{m-1}|_{\text{range } T} = 0$. Inductively on $m$, we can assume that $\text{rk } T|_{\text{range } T} \leq \frac{(m-2)\text{rk } T}{(m-1)}$. The rank-nullity theorem for $T|_{\text{range } T}$ then implies that

$$\dim \text{null}(T|_{\text{range } T}) \geq \text{rk } T - \frac{(m-2)\text{rk } T}{(m-1)} = \frac{\text{rk } T}{m-1}.$$  

Then, the rank-nullity theorem for $T$ shows that

$$\dim V = \dim \text{null } T + \text{rk } T \geq \dim \text{null } T|_{\text{range } T} + \text{rk } T \geq \frac{\text{rk } T}{m-1} + \text{rk } T = \frac{m}{m-1} \text{rk } T.$$  

Hence, $\text{rk } T \leq \frac{(m-1)\dim V}{m}$, as desired.

(9) Recall the fundamental theorem of algebra: that every polynomial $p \in \mathcal{P}_m(\mathbb{C})$ splits as a product of linear factors, i.e., factors of the form $(ax + b)$.

Write $p(x) = a_m x^m + \cdots + a_0 = a_m (x - r_1) \cdots (x - r_m)$, for $r_1, \ldots, r_m \in \mathbb{C}$ (and $a_m \neq 0$).

Suppose that, for $T \in \mathcal{L}(V)$ (with $V$ nonzero), we have $p(T) = 0$, which means, by definition, that the linear transformation $a_m T^m + \cdots + a_1 T + a_0 I$ is the zero transformation. Prove that, for some $r_i$, the transformation $(T - r_i I)$ is not injective. (Use Ch. 3 # 6, which was on last week’s problem set. You don’t have to reprove it.)

A: Note that $p(T) = a_m (T - r_1 I) \cdots (T - r_m I)$, since if we multiply this out, we get the same polynomial in $T$ as if we multiply out $a_m (x - r_1) \cdots (x - r_m)$ (where by a polynomial in $T$, we mean multiplying the constant term by the identity matrix). Hence, $p(T) = 0$ implies that $a_m (T - r_1 I) \cdots (T - r_m I) = 0$. Since this is zero, and $V$ is nonzero (note: I did forget to insert this condition in the original assignment), in particular it is not injective. But we know that the product of injective linear transformations is injective (this was on a previous homework, or it is not difficult to see). So one of these factors $(T - r_1 I), \ldots, (T - r_m I)$ is not injective (note that $a_m \neq 0$ by assumption: I also forgot to say this in the original problem).

(10) Suppose that $T^m = I$, the identity matrix. Show that, for some $m$-th root of unity $\lambda$ (i.e., a complex number such that $\lambda^m = 1$, or equivalently a complex number such that $\lambda = e^{2\pi i k}$ for some integer $k$, which we can take to be $0 \leq k \leq m - 1$), the transformation $(T - \lambda I)$ is not injective.

A: Consider the polynomial $p(x) = x^m - 1$. It splits into linear factors by the fundamental theorem of algebra, i.e., $x^m - 1 = (x - r_1) \cdots (x - r_m)$. The $r_1, \ldots, r_m$ that occur are exactly
the roots of \( p(x) \), i.e., the numbers such that \( p(r_i) = 0 \). But \( p(r) = 0 \) if and only if \( r^m = 1 \), i.e., if and only if \( r \) is an \( m \)-th root of unity. So the \( r_1, \ldots, r_m \) must consist of exactly the \( m \)-th roots of unity. Then, by the previous problem, for some \( r_i \) and hence some \( m \)-th root of unity, \( (T - r_iI) \) is not injective.

Remark: The \( m \)-th roots of unity are \( e^{\frac{2\pi ki}{m}} \) for \( 0 \leq k \leq m - 1 \); there are \( m \) of these, so we can in fact conclude that each occurs once among \( r_1, \ldots, r_m \). This means that we may take \( r_k = e^{\frac{2\pi ki}{m}} \) if we like.