(1) (a) Let $V$ and $W$ be finite dimensional vector spaces and $T: V \to W$ a linear transformation. Prove that there exist bases $(v_1, \ldots, v_n)$ of $V$ and $(w_1, \ldots, w_m)$ of $W$ such that $\mathcal{M}(T, (v_1, \ldots, v_n), (w_1, \ldots, w_m))$ (NOTE: A previous version of the homework used instead the notation $\mathcal{M}_{(v_1, \ldots, v_n), (w_1, \ldots, w_m)}(T)$ has the block form

\[
\mathcal{M}(T, (v_1, \ldots, v_n), (w_1, \ldots, w_m)) = \begin{pmatrix} I_k & 0_{k,n-k} \\ 0_{m-k,k} & 0_{m-k,n-k} \end{pmatrix},
\]

where $I$ is a $k \times k$-identity matrix, and each $0_{a,b}$ refers to a $a \times b$-zero matrix.

(b) Moreover, prove that $k = \text{rk}(T)$.

(2) How to pass between linear transformations and matrices in the presence of bases: Fix the bases $(v_1, \ldots, v_n)$ and $(w_1, \ldots, w_m)$ of $V$ and $W$. Suppose we are given any $m \times n$-matrix $A = (a_{i,j}) \in \text{Mat}(m, n, \mathbb{F})$ (here, as usual, $\text{Mat}(m, n, \mathbb{F})$ denotes the vector space of all $m$ by $n$ matrices, and $A \in \text{Mat}(m, n, \mathbb{F})$ means that $A$ is an $m \times n$ matrix). Let $T_A \in \mathcal{L}(V, W)$ be the unique linear transformation such that

\[T_A(v_i) = a_{1,i}w_1 + \cdots + a_{m,i}w_m, \forall 1 \leq i \leq n.\]

That is,

\[T_A(b_1v_1 + \cdots + b_nv_n) = \sum_{i=1}^{n} \sum_{j=1}^{m} b_ia_{j,i}w_j.\]

Prove the relations:

\[\mathcal{M}(T_A, (v_1, \ldots, v_n), (w_1, \ldots, w_m)) = A, \quad T_{\mathcal{M}(T_A, (v_1, \ldots, v_n), (w_1, \ldots, w_m))} = T_A.
\]

Deduce that the following two maps are inverse to each other:

(a) $\mathcal{L}(V, W) \to \text{Mat}(m, n, \mathbb{F})$ defined by, for all $T \in \mathcal{L}(V, W)$, $T \mapsto \mathcal{M}(T)$ (here “$\mapsto$” is different from $\to$ in that the former is on elements and the latter specifies the source and target: see Muddy Card Responses, Lecture 8, question (1));

(b) $\text{Mat}(m, n, \mathbb{F}) \to \mathcal{L}(V, W)$, defined by $A \mapsto T_A$.

Why does this prove that $\dim \mathcal{L}(V, W) = \dim V \cdot \dim W$, i.e., what basis does one obtain of $\mathcal{L}(V, W)$ from the above? Hint: Consider the image of the basis of $\text{Mat}(m, n, \mathbb{F})$ by matrices with a single entry of one and all other entries zero: there are $mn = \dim V \cdot \dim W$ of these.

(3) Change-of-basis formula:

(a) Prove that, if $T: V \to W$ is a linear transformation between finite dimensional vector spaces, $(v_1, \ldots, v_n)$ is a basis of $V$, $(w_1, \ldots, w_m)$ is a basis of $W$, and $S_V: V \to V$ and $S_W: W \to W$ are invertible linear transformations, then

\[\mathcal{M}(T, (S_Vv_1, \ldots, S_Vv_n), (S_Ww_1, \ldots, S_Ww_m)) = \mathcal{M}(S_W^{-1}TS_V, (v_1, \ldots, v_n), (w_1, \ldots, w_m)).\]

(b) Prove the following general identity (which generalizes Proposition 3.14): Given linear transformations $S: U \to V$ and $T: V \to W$ between finite dimensional vector spaces $U, V,$ and $W$ with bases $(u_1, \ldots, u_p)$, $(v_1, \ldots, v_n)$, and $(w_1, \ldots, w_m)$,

\[\mathcal{M}(TS, (u_1, \ldots, u_p), (w_1, \ldots, w_m)) = \mathcal{M}(T, (v_1, \ldots, v_n), (w_1, \ldots, w_m))\mathcal{M}(S, (u_1, \ldots, u_p), (v_1, \ldots, v_n)).\]
(c) Now, we reinterpret the result of part (a) in terms of matrices. You should use problem (2) and part (b). Let $A \in \text{Mat}(m, n, F)$, $B \in \text{Mat}(n, n, F)$, and $C \in \text{Mat}(m, m, F)$. We perform a change-of-basis on $A$ by changing the basis of the domain (column vectors of length $n$, i.e., $\text{Mat}(n, 1, F)$) by the matrix $B$, and changing the basis of the target (column vectors of length $m$, i.e., $\text{Mat}(m, 1, F)$) by the matrix $C$. Prove that

$$\mathcal{M}(T_A, (T_Bv_1, \ldots, T_Bv_n), (T_Cw_1, \ldots, T_Cw_m)) = C^{-1}AB.$$ 

(4) Suppose $V_1, V_2 \subseteq V$ are subspaces such that $V_1 + V_2 = V$, and $T : V \to W$ is a linear transformation. Prove that

$$\text{rk}(T) = \text{rk}(T|_{V_1}) + \text{rk}(T|_{V_2}) - \dim(T(V_1) \cap T(V_2)).$$

In particular, deduce that $\text{rk}(T) \leq \text{rk}(T|_{V_1}) + \text{rk}(T|_{V_2})$.

(6) Page 60, Exercise 10.
(7) Page 60, Exercise 11.
(8) Page 61, Exercise 22.
(9) Page 61, Exercise 24.