18.700 FALL 2011 MIDTERM 2 SOLUTIONS

TRAVIS SCHEDLER

(1) Definitions (5 points each)
(i) An inner product space
A: A vector space $V$ over $F = \mathbb{R}$ or $\mathbb{C}$, equipped with a pairing $\langle -, - \rangle : V \times V \to F$ satisfying:
- $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle, \forall u, v, w \in V$;
- $\langle au, v \rangle = a \langle u, v \rangle, \forall u, v \in V, a \in F$;
- $\langle u, v \rangle = \langle v, u \rangle, \forall u, v \in V$;
- $\langle v, v \rangle \geq 0$ for all $v \in V$;
- $\langle v, v \rangle = 0$ if and only if $v = 0$.

(ii) The adjoint of a linear transformation
A: For $T \in \mathcal{L}(V, W)$ with $V, W$ inner product spaces, this is a map $T^* \in \mathcal{L}(W, V)$ such that $\langle Tv, w \rangle = \langle v, T^*w \rangle$ for all $v \in V$. (We saw in class and in Axler that, if it exists, then $T^*$ is unique.)

(iii) An orthonormal list
A: This is a list $(e_1, \ldots, e_k)$ of vectors such that $\langle e_i, e_j \rangle = \delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$

(iv) A polar decomposition of a linear transformation
Given $T \in \mathcal{L}(V)$ for $V$ an inner product space, this is an expression $T = SP$ where $S \in \mathcal{L}(V)$ is an isometry and $P \in \mathcal{L}(V)$ is positive.

(v) Give a clear pictorial definition of a block upper-triangular matrix with $1 \times 1$ and $2 \times 2$ diagonal blocks (i.e., draw the matrix and explain what form it must have).
A: For you to do!

(vi) A generalized eigenvector of a linear transformation $T$
A: This is a vector $v$ such that, for some $\lambda \in F$ and some $m \geq 1$, $(T - \lambda I)^m v = 0$.

(2) True/false (10 points each): 5 points for correct answer. If true, give an explanation (5 points, which can refer to results from the book or class); if false, give a counterexample (5 points).

(i) If $F = \mathbb{R}$ and $T \in \mathcal{L}(V)$ admits an orthonormal eigenbasis (for $V$ a finite-dimensional inner product space), then $T$ is self-adjoint.
A: True: this is half of the spectral theorem for real self-adjoint operators; alternatively, if $(e_1, \ldots, e_n)$ is an orthonormal eigenbasis of $T$ with eigenvalues $\lambda_1, \ldots, \lambda_n$, then $\langle Te_i, e_j \rangle = \lambda_i \delta_{ij} = \lambda_j \delta_{ij} = \langle e_i, Te_j \rangle$ for all $e_i$ and $e_j$. Now using the axioms of the inner product (real linearity on the right and also on the left) we get that $\langle Tv, w \rangle = \langle v, Tw \rangle$ for all $v, w \in V$, so $T = T^*$.

(ii) If a linear transformation $T$ on a finite-dimensional inner product space is positive, there is a unique linear transformation $S$ such that $S^2 = T$.
A: False: this was only true if we require that $S$ also be positive. For example, if $T = I$, then we could also have $S = -I$, and then $S^2 = I$ (and $-I \neq I$).

(iii) If a linear transformation $T$ has a nonzero generalized eigenvector of eigenvalue $\lambda$, then it also has a nonzero eigenvector of eigenvalue $\lambda$.
A: True: Suppose that $(T - \lambda I)^m v = 0$ (with $v \neq 0$), and let us assume that $m \geq 1$ is minimal for this property. Then $(T - \lambda I)^{m-1} v \neq 0$ and this is an eigenvector of eigenvalue $\lambda$. 

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(3) Computations (10 points each) (Show your work):

(i) Consider the inner product space \( \mathbb{R}^3 \) with the usual dot product. Perform Gram-Schmidt orthogonalization on the list \( ((1, 1, 1), (-2, 1, -2), (2, 3, -2)) \).

A: First, \( e_1 = (1, 1, 1)/|(1, 1, 1)| = \frac{1}{\sqrt{3}}(1, 1, 1) \). Next, \( e_2' = (-2, 1, -2) - \langle -2, 1, -2, -\frac{1}{\sqrt{3}}(1, 1, 1) \rangle \frac{1}{\sqrt{3}}(1, 1, 1) = (-2, 1, -2) + (1, 1, 1) = (-1, 2, -1) \). So \( e_2 = e_2'/\|e_2\| = \frac{1}{\sqrt{6}}(-1, 2, -1) \). Finally, \( e_3' = (2, 3, -2) - \frac{1}{3}\langle (2, 3, -2), (1, 1, 1) \rangle (1, 1, 1) - \frac{1}{5}\langle (2, 3, -2), (-1, 2, -1) \rangle (-1, 2, -1) = (2, 3, -2) - (1, 1, 1) - (-1, 2, -1) = (2, 0, -2) \). So \( e_3 = e_3'/\|e_3\| = \frac{1}{\sqrt{2}}(1, 0, -1) \). We get:

\[
(e_1, e_2, e_3) = \left( \frac{1}{\sqrt{3}}(1, 1, 1), \frac{1}{\sqrt{6}}(-1, 2, -1), \frac{1}{\sqrt{2}}(1, 0, -1) \right).
\]

(ii) Let \( F = \mathbb{R} \). Find the polar decomposition of the linear transformation \( T_A \) where

\[
A = \begin{pmatrix} 6 & 12 \\ -8 & 9 \end{pmatrix}
\]

Recall that \( T_A \) is the transformation \( T_A(v) = Av \) for \( v \in \text{Mat}(\mathbb{R}, 2, 1) \).

\textbf{Hint:} First find its singular value decomposition. Partial credit will be awarded if this is done (mostly) correctly.

A: To find the singular value decomposition: we first find an eigenbasis \( (e_1, \ldots, e_n) \) of \( \overline{A}A = \begin{pmatrix} 6 & -8 \\ 12 & 9 \end{pmatrix} \begin{pmatrix} 6 & 12 \\ -8 & 9 \end{pmatrix} = \begin{pmatrix} 100 & 225 \\ 0 & 225 \end{pmatrix} \). An eigenbasis is evidently \( e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \). The eigenvalues of \( \overline{A}A \) are \( s_1^2 = 100 \) and \( s_2^2 = 225 \), so the singular values are \( s_1 = 10 \) and \( s_2 = 15 \). Then, \( f_1 = s_1^{-1}Ae_1 = \begin{pmatrix} 3/5 \\ -4/5 \end{pmatrix} \) and \( f_2 = s_2^{-1}Ae_2 = \begin{pmatrix} 4/5 \\ 3/5 \end{pmatrix} \).

Now to write \( A = SP \) for \( S \) an isometry and \( P \) positive, we have \( Pe_i = s_ie_i \) and \( Se_i = f_i \). So since \( (e_1, e_2) \) is the standard basis, we conclude that

\[
S = \begin{pmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{pmatrix}, \quad P = \begin{pmatrix} 10 & 0 \\ 0 & 15 \end{pmatrix}.
\]

(iii) Find the eigenvalues of the matrix

\[
\begin{pmatrix} 2 & 3 & 1 & 1 \\ 1 & 2 & 5 & -7 \\ 0 & 0 & -3 & 5 \\ 0 & 0 & 0 & -3 \end{pmatrix}
\]

Bonus: Find its characteristic and minimal polynomials (up to 5 points for the correct answer and 5 points for the explanation). You can leave them factored if you like. For this, you can work over \( \mathbb{C} \) if you like.

A: This is a block upper-triangular matrix, so the eigenvalues are the same as the eigenvalues of the diagonal blocks. For the upper-left block, these are the roots of its characteristic polynomial, \( x^2 - 4x + 1 \), which are \( 2 \pm \sqrt{3} \). For the other two blocks, the eigenvalues are \( -3 \). So we get \( 2 + \sqrt{3}, 2 - \sqrt{3}, -3 \) as the three eigenvalues.

Bonus: If we rewrite the upper-left block as an upper-triangular (actually diagonal) matrix with complex entries by changing the basis only affecting the first two basis vectors, we get diagonal entries of \( 2 \pm \sqrt{3} \). The resulting matrix is upper-triangular.
with diagonal entries \(2 + \sqrt{3}, 2 - \sqrt{3}, -3, \) and \(-3\). So the characteristic polynomial is, by definition,
\[
\chi_A(x) = (x - (2 + \sqrt{3}))(x - (2 - \sqrt{3}))(x + 3)(x + 3) = (x^2 - 4x + 1)(x + 3)^2.
\]
(Alternatively, one could note that the characteristic polynomial of a block upper-triangular matrix is the same as the product of the characteristic polynomials of the diagonal blocks; we haven’t actually proved this but it follows from the definition in PS 10 \# 10. See PS11.)

Next, let us compute the minimal polynomial. Let \(V = \text{Mat}(C, 4, 1)\). We know that the dimension of \(V(-3)\) is two, since \(-3\) occurs twice; also there are two other one-dimensional eigenspaces. Hence the decomposition theorem shows that \(V = V(2 + \sqrt{3}) \oplus V(2 - \sqrt{3}) \oplus V(-3)\). Then \(p(T) = 0\) if and only if \(p(T)|_{V(2+\sqrt{3})} = p(T)|_{V(2-\sqrt{3})} = p(T)|_{V(-3)} = 0\). The first two happen if and only if \(2 + \sqrt{3}\) and \(2 - \sqrt{3}\) are roots of \(p(x)\).

The last condition happens if and only if \((x + 3)^2 \mid p(x)\), since \((T + 3I)^2|_{V(-3)} = 0\), but \((T + 3I)|_{V(-3)} \neq 0\), and \((T - \lambda I)|_{V(-3)}\) is an isomorphism for a complex number \(\lambda \neq 3\). So the polynomials \(q(x)\) such that \(q(T) = 0\) are exactly the multiples of \(p(x) = \chi_A(x) = (x^2 - 4x + 1)(x + 3)^2\), and this is the minimal polynomial.

Remark: we could have worked over \(\mathbb{R}\) if we preferred, but it makes things slightly more complicated since polynomials don’t factor linearly. We could use the fact that \(\chi_A(A) = 0\) and, by the PS, the minimal polynomial must therefore be a factor of \(\chi_A(x)\), and so then the only one that is zero on \(A\) is \(\chi_A(x)\) itself.

(4) Proofs: 20 points total.

(i) \((10\ \text{points})\) Show that the following are equivalent for \(T \in \mathcal{L}(V)\), with \(V\) a finite-dimensional inner product space:

(a) \(T\) is normal

(b) Every polar decomposition \(T = SP\), with \(S\) an isometry and \(P\) positive, satisfies 
\[SP^2 = P^2S,\]

(c) Some polar decomposition \(T = SP\), with \(S\) an isometry and \(P\) positive, satisfies 
\[SP^2 = P^2S.\]

(Caution: in part (b), the statement does not assume that there exists a polar decomposition. Refer to main theorem(s).)

Hint: You might try \((a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a)\).

Note: We will see that actually \(SP = PS\) in the next problem.

A: \((a) \Rightarrow (b)\): If \(T\) is normal, then \(T^*T = TT^*\), so if \(T = SP\) as well with \(S\) an isometry and \(P\) positive, then \((SP)^*(SP) = P^*S^*SP = P^2\) since \(P = P^*\) and \(S^*S = I\), and this must equal \((SP)^*(SP) = SP^2S^* = SP^2S^{-1}\). So \(P^2 = SP^2S^{-1}\) and hence \(P^2S = SP^2\).

(b) \(\Rightarrow (c)\): By our theorem from the book and class, there always exists a polar decomposition \(T = SP\). So then there is one, and by (b), it satisfies \(SP^2 = P^2S\).

(c) \(\Rightarrow (a)\): If \(T = SP\) and \(SP^2 = P^2S\), then \(T^*T = (SP)^*(SP) = P^*(S^*SP) = P^2\), and \(TT^* = SP^2S^* = SP^2S^{-1}\) again, and now the statement \(SP^2 = P^2S\) implies that \(T^*T = TT^*\).

(ii) \((10\ \text{points})\) Let \(V\) be a finite-dimensional inner product space, \(P \in \mathcal{L}(V)\) be positive, and \(T \in \mathcal{L}(V)\) be arbitrary. Show that the following are equivalent:

(a) \(TP^2 = P^2T\);

(b) For every eigenspace \(U = \text{null}(P^2 - \lambda I)\) of \(P^2\), \(T(U) \subseteq U\). (Hint: Show that this is equivalent to \(TP^2u = P^2Tu\) for every \(u \in U\); and recall the spectral theorem for positive operators.)

(c) \(TP = PT\) (Hint: Show that the eigenspaces of \(P\) and \(P^2\) are the same, with the latter having the square of the eigenvalues of the former.)
Hint: You might try $(a) \iff (b)$, and then show that $(b) \iff (c)$ as indicated.

A: For $(a) \Rightarrow (b)$, let us consider what it means to have $TP^2|_U = P^2T|_U$ for $U = \text{null}(P^2 - \lambda I)$ for some (necessarily nonnegative) eigenvalue $\lambda$ of $P$. This means that, for all $u \in U$, $T(\lambda u) = P^2(Tu)$, i.e., $Tu$ is also an eigenvector of $P^2$ of eigenvalue $\lambda$. Hence $T(U) \subseteq U$.

For the reverse implication, if $T(U) \subseteq U$, we saw that this means $TP^2|_U = P^2T|_U$. Since $P^2$ is positive, it is a direct sum of its eigenspaces, and hence this implies that $TP^2 = P^2T$.

For $(b) \iff (c)$, we can apply the same argument as above with $P$ instead of $P^2$, and conclude that $TP = PT$ if and only if $T(\text{null}(P - \mu I)) \subseteq \text{null}(P - \mu I)$ for all eigenvalues $\mu$ of $P$. But, we know that $P$ has an eigenbasis with nonnegative eigenvalues $\mu \geq 0$, so for each such eigenvalue, $\text{null}(P - \mu I) = \text{null}(P^2 - \mu^2 I)$ (see also the construction of the square root from Axler). More precisely, clearly $\text{null}(P^2 - \mu^2 I) \subseteq \text{null}(P - \mu I)$, but since $V$ is a direct sum of all the $\text{null}(P^2 - \lambda I)$, and for each there is a unique $\mu = \sqrt{\lambda} \geq 0$ such that $\mu^2 = \lambda$, it must be that this inclusion is an equality.

Hence, $T(\text{null}(P - \mu I)) \subseteq \text{null}(P - \mu I)$ for all $\mu$ if and only if $T(\text{null}(P^2 - \lambda I)) \subseteq \text{null}(P^2 - \lambda I)$ for all $\lambda$, i.e., $(b) \Rightarrow (c)$ as well as $(b) \Rightarrow (a)$. 