Lecture 7: Examples of linear operators, null space and range, and the rank-nullity theorem (1)

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Goals (2)
• Understand dimension and infinite-dimensionality
• Dimension formula, finish Chapter 2
• Introduce linear operators
• Null space and range of linear operators

Reading for next time: Finish Chapter 3. Pay attention to isomorphisms and matrices.

Warm-up exercise (3)
Suppose $V = V_1 \oplus V_2$, and $W$ is another vs.

(a) Given any linear maps $T_1 : V_1 \rightarrow W$ and $T_2 : V_2 \rightarrow W$, prove that there is a unique linear map $T : V \rightarrow W$ such that, for all $v_1 \in V_1$ and $v_2 \in V_2$, $T(v_1 + v_2) = T_1(v_1) + T_2(v_2)$.
Call this linear map $T_1 \oplus T_2$.

(b) For every linear map $T : V \rightarrow W$, let $T_1 : V_1 \rightarrow W$ and $T_2 : V_2 \rightarrow W$ be the restrictions. Prove that $T = T_1 \oplus T_2$.

Conclusion: Linear maps $T : V \rightarrow W$ are the same as pairs $(T_1, T_2)$ of linear maps $T_1 : V_1 \rightarrow W$ and $T_2 : V_2 \rightarrow W$.

Solution to warm-up exercise (4)

(a) Well-definition: For every $v \in V$, there exist unique $v_1 \in V_1$ and $v_2 \in V_2$ such that $v = v_1 + v_2$, so we can define $T$ by $T(v) = T_1(v_1) + T_2(v_2)$. 

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Linearity: We first show that \( T(u + v) = T(u) + T(v) \). Write \( u = u_1 + u_2 \) and \( v = v_1 + v_2 \) for \( u_1, v_1 \in V_1 \) and \( u_2, v_2 \in V_2 \). Then,

\[
T(u + v) = T((u_1 + v_1) + (u_2 + v_2)) = T_1(u_1 + v_1) + T_2(u_2 + v_2) \\
= (T_1(u_1) + T_1(v_1)) + (T_2(u_2) + T_2(v_2)) \\
= (T_1(u_1) + T_2(u_2)) + (T_1(v_1) + T_2(v_2)) = T(u) + T(v).
\]

Similarly, \( T(av) = T(a(v_1 + v_2)) = T_1(av_1) + T_2(av_2) = a(T_1(v_1) + T_2(v_2)) = aT(v) \).

(b) For every \( v \in V \), write \( v = v_1 + v_2 \) for \( v_i \in V_i \). Then \( T(v) = T(v_1 + v_2) = T(v_1) + T(v_2) = T_1(v_1) + T_2(v_2) = (T_1 \oplus T_2)(v) \).

**Linear transformations (review) (5)**

**Definition 1.** Given two vector spaces \( V \) and \( W \) over \( F \), a linear transformation \( T : V \to W \) is a function satisfying:

- \( T(u + v) = T(u) + T(v), \forall u, v \in V \);
- \( T(av) = aT(v), \forall a \in F, v \in V \).

**Examples:**

- The identity map \( I : V \to V \) is a linear transformation. So is the zero map \( V \to \{0\} \) (or the zero map \( V \to V \)).
- Scalar multiplication: \( \lambda I : V \to V : \lambda I(v) = \lambda v \) for all \( v \).
- The inclusion \( T : F^m \to F^n \) for \( m \leq n \) as the first coordinates: \( T(a_1, \ldots, a_m) = (a_1, \ldots, a_m, 0, \ldots, 0) \).
- The evaluation map: let \( V = \text{the vs. of functions } \{1, \ldots, n\} \to F \). Then for any \( 1 \leq j \leq n \), we have \( \text{ev}_j : V \to F, \text{ev}_j(f) = f(j) \).

**More sophisticated examples (6)**

- Differentiation of differentiable functions (\( R \) to \( R \)).
  - E.g., \( x^n \mapsto nx^{n-1} \). Gives \( \mathcal{P}(R) \to \mathcal{P}(R) \), linear.
- Integration of continuous real functions (denoted \( \mathcal{C}(R) \)):
  - Definite integration: given \( a \leq b, \ f \mapsto \int_a^b f(y) \, dy \in R \). Gives \( \mathcal{C}(R) \to R \).
  - Indefinite integration: \( f \mapsto \int_a^x f(y) \, dy \), gives \( \mathcal{C}(R) \to \mathcal{C}(R) \).
- Multiplication by a function: Fix a function \( f \). This defines \( M_f : \text{functions } \to \text{functions}, M_f(g) := fg \).
For example, multiplication by $2x^3$: $M_{2x^3}(1 + x) = 2x^3 + 2x^4$.

- Backward shift: $T : \mathbb{F}^\infty \to \mathbb{F}^\infty$, $T(x_1, x_2, x_3, \ldots) = (x_2, x_3, \ldots)$.

- Projections: Given $V = U \oplus W$, we obtain a map $V \to U$ called projection: $(u + w) \mapsto u$.
  
  - Since all $v$ can uniquely be written as $v = u + w$, the map $v \mapsto w$ is well-defined!

Examples of linear transformations $\mathbb{R}^2 \to \mathbb{R}^2$ (7)

- Rotations $R_\theta$ (counterclockwise). E.g., the $90^\circ$ rotation, $R_{90^\circ}(x, y) = (-y, x)$.

- Reflections. E.g., about the $x$-axis, $(x, y) \mapsto (x, -y)$.

- Question: What is the reflection about the $x = y$ axis? Answer: $(x, y) \mapsto (y, x)$.

Matrices (8)

given any two-by-two matrix,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

(0.1)

this defines a linear map $T_A : \mathbb{F}^2 \to \mathbb{F}^2$:

- View $v = (x, y)$ as the column vector $\begin{pmatrix} x \\ y \end{pmatrix}$.

- Then, $T_A(v)$ is defined by the product $Av$, i.e.,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}.$$  

(0.2)

- Thus, $T_A(v) = (ax + by, cx + dy)$.

**Theorem 2** (Exercise: generalizes Chap 3, problem 1 on PSet). All linear maps $\mathbb{F}^2 \to \mathbb{F}^2$ are of the form $T_A$ for a unique matrix $A$. This matrix is defined by

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $T_A(1, 0) = (a, c)$ and $T_A(0, 1) = (b, d)$.

**Null space (9)**

**Definition 3.** The null space of a linear transformation $T : V \to W$ is $\text{null } T := \{ v \in V \mid T(v) = 0 \} \subseteq V$.

Exercise: The null space is a subspace.

Examples:
• The null space of the backwards shift, $T : \mathbf{F}^\infty \to \mathbf{F}^\infty$, given by $T(a_0, a_1, a_2, \ldots) = (a_1, a_2, \ldots)$ is: $\{ (a, 0, 0, 0, \ldots) \mid a \in \mathbf{F} \}$.

• The null space of the identity, $I : V \to V$, is: $\{ 0 \}$.

• The null space of the zero map, $0 : V \to V$, is: $V$.

• The null space of a rotation of $\mathbf{R}^2$ is: $\{ 0 \}$.

• The null space of the orthogonal (=perpendicular) projection $\mathbf{R}^2 \to x$ - axis is: $y$-axis.

**Proposition 3.2: Injectivity and null space (10)**

Recall:

**Definition 4.** A function $f : X \to Y$ is injective if, for every pair $x_1 \neq x_2$ of elements of $X$, $f(x_1) \neq f(x_2)$.

**Proposition 0.3** (Proposition 3.2). A linear transformation $T : V \to W$ is injective if and only if $\text{null } T = \{ 0 \}$.

**Proof (on board).** Suppose $T$ is injective. Since $T(0) = 0$, if $T(v) = 0$, then $v = 0$. Hence, $v \in \text{null } T$ implies $v = 0$, i.e., $\text{null } T = \{ 0 \}$.

Conversely, suppose that $\text{null } T = \{ 0 \}$. If $T(v) = T(w)$, then $T(v - w) = 0$, and hence $v - w = 0$. So $v = w$. Hence, $T(v) = T(w)$ implies $v = w$, i.e., $T$ is injective. □

**Range (11)**

**Definition 5.** The range of a function $f : X \to Y$ is the subset range $f := \{ f(x) \mid x \in X \} \subseteq Y$.

**Definition 6.** A function $f : X \to Y$ is surjective if (range $f$) = $Y$.

Exercise: If $T : V \to W$ is a linear map, then (range $T$) $\subseteq W$ is a subspace.

**Rank-nullity theorem (12)**

**Theorem 7** (Theorem 3.4; “Rank-nullity theorem”). If $V$ is f.d. and $T : V \to W$ is linear, then

$$\dim V = \dim \text{null}(T) + \dim \text{range}(T).$$

Definition: the rank of $T$ is $\text{rk } T := \dim \text{range}(T)$.

**Corollary 8.** $\text{rk } T \leq \min\{ \dim V, \dim W \}$. Moreover, $\text{rk } T = \dim V$ if and only if $T$ is injective, and $\text{rk } T = \dim W$ if and only if $T$ is surjective.

**Corollary 9** (Corollaries 3.5 and 3.6). Let $T \in \mathcal{L}(V, W)$ with $V$ and $W$ finite-dimensional.

(i) If $\dim V > \dim W$, then $T$ is not injective.

(ii) If $\dim V < \dim W$, then $T$ is not surjective.
Proof of rank-nullity theorem (on board) (13)
Recall: $T : V \to W$ is linear, and $V$ is f.d.

- Let $(u_1, \ldots, u_m)$ be a basis of null $T$.
- Extend $(u_1, \ldots, u_m)$ to a basis $(u_1, \ldots, u_m, v_1, \ldots, v_n)$ of $V$.
- Then, dim $V = m + n = \dim \text{null } T + n$. We need: dim $\text{range } T = n$.
- Claim: $(T(v_1), \ldots, T(v_n))$ is a basis for $\text{range } T$. This proves the theorem.

Proof of Claim (on board) (14)
Claim: $(T(v_1), \ldots, T(v_n))$ is a basis for $\text{range } T$.

- Spanning: First, $\text{Span}(T(u_1), \ldots, T(u_m), T(v_1), \ldots, T(v_n)) = T(\text{Span}(u_1, \ldots, u_m, v_1, \ldots, v_n)) = T(V)$, by linearity of $T$.
- But, $T(u_1) = \cdots = T(u_m) = 0$ since $u_1, \ldots, u_m \in \text{null}(T)$.
- Hence, $(T(v_1), \ldots, T(v_n))$ span $T(V) = \text{range}(T)$.
- Lin. ind.: Suppose $a_1 T(v_1) + \cdots + a_n T(v_n) = 0$.
- Then, $T(a_1 v_1 + \cdots + a_n v_n) = 0$.
- So $a_1 v_1 + \cdots + a_n v_n \in \text{null } T$. We can therefore write
  $$a_1 v_1 + \cdots + a_n v_n = b_1 u_1 + \cdots + b_m u_m.$$  
- By linear independence of $(u_1, \ldots, u_m, v_1, \ldots, v_n)$, this implies that $a_1 = \cdots = a_n = 0 = b_1 = \cdots = b_m$. So, $(T(v_1), \ldots, T(v_n))$ is lin. ind.