Lecture 16: Adjoints, self-adjoint operators, and the spectral theorem for self-adjoint operators (1)

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Goals (2)

• Recap: orthogonal projection and approximation
• Adjoints
• Self-adjoint operators
• The spectral theorem for self-adjoint operators

Read Chapter 7 and do PS 8.

Warm-up exercise (3)

Consider \( P_2(\mathbb{C}) \) with the inner product \( \langle f, g \rangle = \int_0^1 f(x)\overline{g(x)} \, dx \). Use Gram-Schmidt orthogonalization on the list \((x,1)\) to find an orthonormal basis. (This won’t be the same as the one for \((1, x)\))!

Solution to warm-up exercise (4)

• Let \((v_1, v_2) = (x, 1)\); we wish to find \((e_1, e_2)\) orthonormal with \(\text{Span}(v_1, \ldots, v_k) = \text{Span}(e_1, \ldots, e_k)\) for \(k \in \{1, 2\}\).

• First \(\int_0^1 (x)^2 \, dx = \frac{1}{3}\) so \(e_1 := \sqrt{3} \cdot x\) has norm one.

• Then \(\langle 1, e_1 \rangle = \int_0^1 \sqrt{3} \cdot x \, dx = \frac{\sqrt{3}}{2}\), so \(e'_2 = 1 - \frac{\sqrt{3}}{2} \cdot \sqrt{3} \cdot x = 1 - \frac{3}{2}x\).

• We then set \(e_2 = e'_2 / \|e'_2\|\). Here \(\|e'_2\|^2 = \int_0^1 (1 - \frac{3}{2}x)^2 \, dx = \int_0^1 (1 - 3x + \frac{9}{4}x^2) \, dx = 1 - \frac{3}{2} + \frac{3}{4} = \frac{1}{4}\).

• So \(\|e'_2\| = \frac{1}{2}\). Hence \(e_2 = 2 - 3x\).

We get: \((\sqrt{3} \cdot x, 2 - 3x)\), as our orthonormal list. This is not the same as \((1, -\sqrt{3} + 2\sqrt{3} \cdot x)\), which we got starting with \((1, x)\).
Recap of orthogonal projection (5)

At the end of last class, we proved, for $U$ a finite-dimensional subspace of $V$,

**Proposition 0.1** (Proposition 6.36). Let $v \in V$. Then, for all $u \in U$, $\|v - P_{U,U^\perp}(v)\| \leq \|v - u\|$.

This says that $P_{U,U^\perp}(v)$ is the closest approximation of $v$ in the subspace $U \subseteq V$.

Moreover, we gave a formula for $P_{U,U^\perp}$: for $e_1, \ldots, e_m$ an orthonormal basis of $U$,

$$P_{U,U^\perp}(v) = \langle v, e_1 \rangle e_1 + \cdots + \langle v, e_m \rangle e_m.$$ 

We gave the example (Axler pp. 114–6): $V =$ the space of continuous functions on $[-\pi, \pi]$ with inner product $\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x) dx$, $U$ is the vector space of polynomials of degree $\leq 5$.

Then $P_{U,U^\perp} \sin x = 0.9879x - 0.1552x^3 + 0.0056x^5$, and the graph of $\sin x$ was indistinguishable from this on $[-\pi, \pi]$.

Application of warm-up exercise (6)

As an application of the warm-up exercise, we can let $V =$ continuous functions on $[0,1]$ with $\langle f, g \rangle = \int_0^1 f(x)g(x) dx$, and $U \subseteq V$ the subspace of polynomials of degree $\leq 1$.

Then, the projection of any continuous function $f(x)$ on $[0,1]$ onto linear polynomials is:

$$P_{U,U^\perp}(f)(x) = \langle f(x), \sqrt{3} \cdot x \rangle \sqrt{3} \cdot x + \langle f(x), 2 - 3x \rangle (2 - 3x)$$

$$= 3x \int_0^1 xf(x) \, dx + (2 - 3x) \int_0^1 (2 - 3x)f(x) \, dx.$$ 

The approximation from Axler of $\sin x$ is obtained in the same manner (with more integrals).

Exercise: Check that $P_{U,U^\perp}$ is the same if one uses the other orthonormal basis we got, $(1, -\sqrt{3} + 2\sqrt{3} \cdot x)$. Indeed, $P_{U,U^\perp}$ does not depend on the orthonormal basis chosen!

Adjoint operators (7)

Let $T \in \mathcal{L}(V,W)$, where $V$ and $W$ are inner product spaces.

**Definition 1.** An *adjoint* operator $T^* \in \mathcal{L}(W,V)$ is one such that

$$\langle Tv, w \rangle = \langle v, T^* w \rangle, \quad \forall v \in V, w \in W. \quad (0.2)$$

**Lemma 2.** If $T^*$ exists, it is unique.

**Proof (on board).** \hspace{0.5cm} $\bullet$ Suppose that $S_1, S_2$ are two adjoints, so $\langle v, S_1 w \rangle = \langle Tv, w \rangle = \langle v, S_2 w \rangle$ for all $v \in V$ and $w \in W$. 

2
• Then \( \langle v, (S_1 - S_2)w \rangle = 0 \) for all \( v \in V \) and \( w \in W \).

• Hence, for all \( w \in W \), \( \langle (S_1 - S_2)w, (S_1 - S_2)w \rangle = 0 \), implying \( (S_1 - S_2)w = 0 \). So \( S_1 = S_2 \).

Existence of adjoints \((V \text{ f.d.}) \) (8)

**Proposition 0.3.** If \( V \) is f.d., then for all \( T \in \mathcal{L}(V, W) \), there exists a (unique) adjoint \( T^* \in \mathcal{L}(W, V) \).

**Proof (on board).**

• Let \( (e_1, \ldots, e_n) \) be an orthonormal basis of \( V \).

• Define \( T^* \) by \( T^* w = \sum_{j=1}^{n} \langle Te_j, w \rangle e_j \).

• By additivity and homogeneity, \( T^* \) is linear.

• \( \langle e_j, T^* w \rangle = \langle Te_j, w \rangle \) for all \( j \).

• By additivity and homogeneity, \( \langle v, T^* w \rangle = \langle Tv, w \rangle \) for all \( v, w \). So \( T^* \) is an adjoint.

Properties of adjoints (9)

When \( S \) and \( T \) exist (e.g., their domains are f.d.) and the statements make sense:

• \( (T + S)^* = T^* + S^* \);

• \( (aT)^* = \overline{a} T^* \);

• \( (T^*)^* = T \);

• \( I^* = I \);

• \( (ST)^* = T^* S^* \), when \( T^* \) and \( S^* \) exist.

Proofs: exercise! E.g.: \( \langle (T + S)v, w \rangle = \langle Tv, w \rangle + \langle Sv, w \rangle = \langle v, T^* w \rangle + \langle v, S^* w \rangle = \langle v, (T^* + S^*) w \rangle \).

We deduce: when \( V \) and \( W \) f.d., then the adjoint map \( \mathcal{L}(V, W) \to \mathcal{L}(W, V) \) is invertible, with the adjoint map \( \mathcal{L}(W, V) \to \mathcal{L}(V, W) \) as the inverse, since \( (T^*)^* = T \) for all \( T \). F.d. is required since otherwise \( T^* \) does not always exist.

Caution: the adjoint map is not linear: \( (aT)^* = \overline{a} T^* \).

Further properties (10)

Let \( T : V \to W \), with \( V, W \) finite-dimensional inner product spaces.

**Proposition 0.4** (Proposition 6.46).

(a) \( \text{null } T^* = (\text{range } T)^\perp \)

(b) \( \text{range } T^* = (\text{null } T)^\perp \)

(c) \( \text{null } T = (\text{range } T^*)^\perp \)
Proof. (a) \( T^* w = 0 \iff \langle v, T^* w \rangle = 0 \) for all \( v \iff \langle Tv, w \rangle = 0 \) for all \( v \iff w \in (\text{range } T)^\perp. \)

(d) Take \( \perp \) of both sides of (a), using \( (U^\perp)^\perp = U. \)

Matrix of operators and adjoints (11)

Let \( (e_1, \ldots, e_n) \) and \( (f_1, \ldots, f_m) \) be orthonormal bases of \( V \) and \( W. \)

Proposition 0.5. Let \( A = (a_{jk}) = \mathcal{M}(T) \) and \( B = (b_{jk}) = \mathcal{M}(T^*). \) Then \( a_{jk} = \langle Te_k, f_j \rangle \) and \( b_{jk} = \langle T^* f_k, e_j \rangle = \overline{a_{kj}}. \) Hence, \( B = \overline{A^t}. \)

Proof (on board).

• The formula for \( A \) follows because \( Te_k = \sum_{j=1}^m \langle Te_k, f_j \rangle f_j. \)

• Then, \( b_{jk} = \overline{a_{kj}} \) is a consequence of the definition of \( T^* \) together with conjugate symmetry.

Self-adjoint operators (12)

Definition 3. An operator \( T \in \mathcal{L}(V) \) is self-adjoint if \( T = T^*. \)

Proposition 0.6 (Proposition 7.1). All eigenvalues of a self-adjoint operator are real.

Proof. Let \( v \in V \) be nonzero such that \( Tv = \lambda v. \) Then, \( \lambda \langle v, v \rangle = \langle Tv, v \rangle = \langle v, Tv \rangle = \overline{\lambda} \langle v, v \rangle. \)

Spectral theorem for self-adjoint operators (13)

From now on, all our vector spaces are finite-dimensional inner product spaces.

Theorem 4 (Theorem 7.13+). \( T \) is self-adjoint iff \( T \) admits an orthonormal eigenbasis with real eigenvalues.

Physical interpretation: Self-adjoint (or “Hermitian”) operators are observable, and the theorem says that every state is a superposition (linear combination) of pure states (eigenvectors) with real observables (eigenvalues).

Proof (on board): for \( F = C: \)

• We already know that \( \mathcal{M}(T) \) is upper-triangular in some orthonormal basis.

• Then, \( T = T^* \) iff the matrix equals its conjugate transpose, i.e., it is diagonal with real values on the diagonal.

• So, \( T = T^* \) if and only if, in some orthonormal basis, \( \mathcal{M}(T) \) is diagonal with real entries, i.e., if and only if there is an orthonormal eigenbasis with real eigenvalues.
Proof of spectral theorem for self-adjoint real operators (14)

Let $F = \mathbb{R}$.

- In some orthonormal basis, the $\mathcal{M}(T)$ is block upper-triangular with $1 \times 1$ and $2 \times 2$ blocks (the latter with no real eigenvalues).
- So $T = T^*$ if and only if $\mathcal{M}(T)$ is symmetric ($\mathcal{M}(T) = \mathcal{M}(T)^t$).
- This is true if and only if it is block diagonal with symmetric $2 \times 2$ blocks.
- However, symmetric matrices have real eigenvalues by Prop. 7.1. In particular a real eigenvalue exists.
- The $2 \times 2$ matrices have no real eigenvalues, so they cannot be symmetric. So there can only be $1 \times 1$ blocks.
- That is, we showed that $T = T^*$ if and only if, in some orthonormal basis $\mathcal{M}(T)$ is diagonal, i.e., there is an orthonormal eigenbasis.