One of these will appear on the actual final (possibly slightly modified or simplified). These are not really a comprehensive review set, as they are weighted towards recent material. The exam will probably have a few more basic things as well. It will have essential definitions from anywhere in the course, and will include some important computations from anywhere particularly recently; see the final review packet for a good list of the possibilities.

1. Generalize the last problem of Midterm 2: Suppose that $T \in L(V)$ satisfies $V = V_1 \oplus \cdots \oplus V_k$ with each $V_i$ an eigenspace of $T$ (so with distinct eigenvalues). ($V$ need not be finite-dimensional; if it is, this is the same as saying that $V$ admits an eigenbasis).

Prove that $S \in L(V)$ satisfies $TS = ST$ if and only if $S(V_i) \subseteq V_i$ for all $i$.

2. Suppose that $S, T \in L(V)$ with $V$ finite-dimensional and $F$ arbitrary. Show that $\text{rk}(S+T) \leq \text{rk}(S) + \text{rk}(T)$, with equality holding only if $\text{range}(S) \cap \text{range}(T) = \{0\}$.

3. (i) If $T : V \to W$ is injective and $W$ is finite-dimensional, prove that there exists $S : W \to V$ such that $ST = I$.

(ii) If $T : V \to W$ is surjective and $V$ is finite-dimensional, prove there exists $S : W \to V$ such that $TS = I$.

4. Compute a Jordan basis of the nilpotent transformation

$$N = \begin{pmatrix}
1 & -5 & 7 \\
-1 & -3 & 5 \\
-1 & -1 & 2
\end{pmatrix}$$

5. Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be the operation $S(x, y) = (y, x)$. Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a transformation such that $ST = TS$ and $T$ has eigenvalues $\lambda_1$ and $\lambda_2$ (not necessarily distinct, but these are all the eigenvalues). Find all possible transformations $T$ (hint: consider (1), although here it's a very special case). How many are there?

6. Sort of generalization of the previous exercise: Suppose $S : \mathbb{R}^n \to \mathbb{R}^n$ has $n$ distinct eigenvalues. Let $TS = ST$ and suppose we know that the eigenvalues of $T$ are $\lambda_1, \ldots, \lambda_n$, this time all distinct (for simplicity!) How many possible $T$ satisfying these properties are there?

7. Suppose that $S, T \in L(V)$ and $V$ is a finite-dimensional inner product space. Let $S$ be normal. Show that $TS^* = S^*T$ if and only if $TS = ST$. (Hint: consider (1)).

8. Now suppose that $S$ and $T$ are both normal (on a f.d. inner product space). Let $F = \mathbb{C}$. Show that the following are equivalent:

   (i) $S + T$ is normal;
   (ii) $ST = TS$;
   (iii) There is an orthonormal basis which is an eigenbasis for $S$ and $T$ at the same time (hence also of $S + T$).

9. Suppose that $S$ and $T$ are normal on a f.d. inner product space. Show that the following are equivalent:

   (i) $\text{rk}(S) + \text{rk}(T) = \text{rk}(S + T)$;
(ii) $V = \text{range}(S) \oplus \text{range}(T) \oplus (\text{null } S \cap \text{null } T)$.

(10) Suppose that $A$ is a block upper-triangular matrix with diagonal blocks $A_1, \ldots, A_k$, and that each diagonal block $A_i$ is upper-triangular with all diagonal entries equal to $\lambda_i$. Furthermore suppose that $\lambda_1, \ldots, \lambda_k$ are all distinct. (In other words, each $A_i$ is the matrix of a transformation with $V = V(\lambda_i)$, and all the $\lambda_i$ are distinct.)

Prove that $A$ is conjugate to the block-diagonal matrix with blocks $A_1, \ldots, A_k$, i.e., setting $T = T_A$, then in some other basis, $M(T_A)$ is block-diagonal with these blocks.

Hint: Show that, for every eigenvalue $\lambda_j$ of $A$, that $\dim \text{null}((A - \lambda_j I)^k) = \dim \text{null}((A_j - \lambda_j I)^k)$, so that the Jordan form of $A$ depends only on $A_1, \ldots, A_k$ and not on what is in the blocks above these diagonal blocks. On the other hand, any two matrices with the same Jordan form are conjugate (they can be brought to the same matrix in different bases).

(11) Find the Jordan form of the matrix

$$\begin{pmatrix}
3 & 7 & 105 & -233 \\
0 & 3 & -142 & 32 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.$$  \hspace{1cm} \text{Hint: by the previous question, it doesn’t matter what is in the upper right two-by-two block (you can also see this explicitly).}

(12) Suppose that $T \in \mathcal{L}(V)$ with $V$ a finite-dimensional inner product space. Suppose that $T$ is a self-adjoint isometry, and that $T$ does not commute with every transformation (i.e., there exists some $S$ such that $TS \neq ST$). Compute the minimal polynomial of $T$.

(13) Let $T \in \mathcal{L}(V)$ for $V$ finite-dimensional and $\mathbf{F}$ arbitrary. Suppose that $\chi_T(x)$ splits into linear factors (not necessarily distinct), i.e., $\chi_T(x) = (x - a_1) \cdots (x - a_n)$ for some $a_1, \ldots, a_n \in \mathbf{F}$. Prove that $V = \bigoplus_{\lambda} V(\lambda)$, for $\lambda \in \mathbf{F}$ the eigenvalues of $T$. [Hint: apply the strengthened decomposition theorem and the Cayley-Hamilton theorem!]

(14) Conclude from the previous question and results in class that the following are equivalent:

- $\chi_T(x)$ splits into linear factors (over $\mathbf{F}$) \textit{(not necessarily distinct)};
- $V = \bigoplus_{\lambda} V(\lambda)$ for $V(\lambda)$ the generalized eigenspaces of $T$ (with $\lambda \in \mathbf{F}$);
- In some basis, $M(T)$ is upper-triangular;
- In some basis, $M(T)$ is in Jordan form.

(15) Deleted...

(16) Compute $e^A$ where $A$ is the Jordan form matrix

$$A = \begin{pmatrix}
2 & 1 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & -1
\end{pmatrix}.$$  \hspace{1cm} \text{Hint: by the previous question, it doesn’t matter what is in the upper right two-by-two block (you can also see this explicitly).}

(17) Let $\mathbf{F} = \mathbf{C}$. Let $J = \lambda I + N_n$ be a Jordan block matrix with $\lambda \neq 0$. Show that, for all $k \geq 1$, the Jordan form of $J^k$ is $\lambda^k I + N_n$, i.e., it is also a single Jordan block, but with $\lambda^k$ on the diagonal.

Hint: This amounts to showing that $\dim \text{null}((J^k - \lambda^k I)^m) = \dim \text{null}((J - \lambda I)^m)$ for all $m \geq 1$ (remark: the latter is just $\min(m, n)$ as we saw in the warm-up to Lecture 22). In fact, we can prove that $\dim \text{null}((J^k - \lambda^k I)^m) = \dim \text{null}((J - \lambda I)^m)$ for all $m \geq 1$. To do so, use that

$$(J^k - \lambda^k I)^m = p_k^m(J), \quad p_k(x) = (x^k - \lambda^k) = (x - \lambda)q_k(x),$$

where $\lambda$ is not a root of $q_k(x)$ (for this last fact use that $\mathbf{F} = \mathbf{C}$, so the roots of $p_k(x)$ are distinct and are $\lambda \cdot \exp(2j\pi i/k)$ for $0 \leq j \leq k - 1$). Since $\lambda$ is not a root of $q_k(x)$, conclude
that \( q_k(J) \) is invertible. Hence \( p^m_k(J) = (J - \lambda I)^m q^m_k(J) \), with \( q^m_k(J) \) invertible. Therefore \( \text{null}(p^m_k(J)) = \text{null}((J - \lambda I)^m) \).

(18) Using the previous problem, prove the following: Suppose that the Jordan form of a transformation \( T \) has no Jordan blocks of eigenvalue zero of size greater than one. Let \( m \geq 1 \) be any positive integer. Show that the Jordan form of \( T^m \) is the same as the Jordan form of \( T \), except with eigenvalues taken to the \( m \)-th power. (Hint: In a Jordan basis of \( T \), \( T^m \) is block diagonal with the blocks equal to the \( m \)-th powers of the Jordan blocks of \( T \); then apply the previous problem).

(19) Suppose that \( F = \mathbb{C} \) and \( T \in \mathcal{L}(V) \) with \( V \) finite-dimensional. Suppose further that \( T \) is invertible and that its minimal polynomial equals its characteristic polynomial. Suppose \( T \) has \( m \) distinct eigenvalues. Using the previous problem, show that there are exactly \( 2^m \) distinct square roots, based on choosing square roots of each eigenvalue. More generally, show that there are exactly \( k^m \) distinct \( k \)-th roots, based on choosing \( k \)-th roots of each eigenvalue.

Hint: remember what we know about the minimal polynomial equalling the characteristic poly in terms of Jordan form!

(20) In fact, the following converse holds: if \( T \in \mathcal{L}(V) \) (with \( V \) f.d. and \( F = \mathbb{C} \)) with \( T \) invertible, and the minimal and characteristic polynomials of \( T \) are unequal, then \( T \) has infinitely many square roots.

In this problem, prove this under the additional assumption that \( V \) is an inner product space and \( T \) is normal. Can you give an explicit example of such a \( T \)?