There are two parts. Please attempt all problems. Look through them first and do the easiest ones first. Some partial credit will be awarded for correct steps towards a solution. Before the test ends, make sure to write something for all the questions you can. You might not need all the paper space. There is an extra sheet of paper at the end of each part. Use scratch paper for your own work, if applicable, and in these sheets include what is to be graded. Be neat!

(1) Let $\mathbb{F}$ be a field. Define the following (5 points each):

(i) A vector space $V$ over $\mathbb{F}$

A vector space is a set $V$ equipped with an addition map $V \times V \to V$ and a scalar multiplication map $\mathbb{F} \times V \to V$ satisfying the following properties:

– (Additive identity): There is a zero vector $0 \in V$, such that $v + 0 = v = 0 + v$ for all $v \in V$;
– (Additive inverse): For all $v \in V$, there is an additive inverse $-v \in V$ such that $v + (-v) = 0 = (-v) + v$;
– (Associativity of addition): For all $u, v, w \in V$, $(u + v) + w = u + (v + w)$.
– (Commutativity of addition): For all $v, w \in V$, $v + w = w + v$.
– (Multiplicative identity): For all $v \in V$, $1 \cdot v = v$.
– (Associativity of multiplication): For all $\lambda, \mu \in \mathbb{F}$, and all $v \in V$, $(\lambda \mu) v = \lambda (\mu v)$ and $\lambda (v + w) = \lambda v + \lambda w$.

Note: in the book, the two associativity properties are grouped together. This is also fine, if you included both of them.

(ii) A linear transformation $T : V \to W$ (given vector spaces $V$ and $W$)

This is a function from $V$ to $W$ satisfying the properties that, for all $u, v \in V$ and all $\lambda \in \mathbb{F}$,

\begin{align*}
T(u + v) &= T(u) + T(v), \\
T(\lambda v) &= \lambda T(v).
\end{align*}

Note that one could combine these into one, and say $T(\lambda u + \mu v) = \lambda T(u) + \mu T(v)$, for example.

(iii) A basis for a vector space $V$ (if you use linear independence or spanning, you must define these too!)

This is a list of vectors $(v_1, \ldots, v_n)$ such that, for all $v \in V$, there exist unique $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$ such that $v = \lambda_1 v_1 + \cdots + \lambda_n v_n$.

Alternatively, this is a linearly independent spanning list. Here, a linearly independent list is a list $(v_1, \ldots, v_k)$ such that $\lambda_1 v_1 + \cdots + \lambda_k v_k = 0$ implies that $\lambda_1 = \cdots = \lambda_k = 0$, and a spanning list is a list $(v_1, \ldots, v_k)$ such that, for all $v \in V$, there exist $\lambda_1, \ldots, \lambda_k$ such that $v = \lambda_1 v_1 + \cdots + \lambda_k v_k$.

(iv) The matrix $\mathcal{M}(T)$ of $T : V \to W$ in terms of bases of $V$ and $W
Let \((v_1,\ldots,v_n)\) be a basis of \(V\) and \((w_1,\ldots,w_m)\) be a basis of \(W\). Write \(T(v_j) = a_{1,j}w_1 + \cdots + a_{m,j}w_m\) for all \(j\), and some \(a_{i,j} \in F\). Then, \(\mathcal{M}(T)\) is the matrix

\[
\mathcal{M}(T) = \begin{pmatrix}
a_{1,1} & \cdots & a_{1,n} \\
\vdots & \ddots & \vdots \\
a_{m,1} & \cdots & a_{m,n}
\end{pmatrix}.
\]

(v) A row echelon form matrix (draw a picture, use more precise wording than I used in class!)

This is a matrix such that the first nonzero entry in each row is a one, and all the entries below and to the left of this are zero. More precisely, let \(A = (a_{ij})\), and suppose that in the \(i\)-th row of \(A\), the first nonzero entry is in the \(p_i\)-th column. Then, \(A\) is in row echelon form if:

(a) \(a_{ip_i} = 1\);

(b) \(a_{kl} = 0\) whenever \(k > i\) and \(\ell \leq p_i\).

I leave the picture to you!

(vi) An eigenvector \(v\) for \(T \in L(V)\) of eigenvalue \(\lambda\)

This is a vector \(v \in V\) such that \(Tv = \lambda v\). (Note: \(\lambda\) is only called an eigenvalue of \(T\) if there exists a nonzero eigenvector \(v\).)

(2) (10 points): What are all of the subspaces of \(\mathbb{R}^2\)? **Give also an explicit definition with set notation!**

The subspaces are \(\{0\}\) (otherwise denoted by 0), \(\mathbb{R}^2\) itself, and all of the lines through the origin, \(L_{\lambda,\mu} = \{(\lambda t, \mu t) : t \in \mathbb{R}\}\), where \(\lambda\) and \(\mu\) are not both zero. (Note that \(L_{\lambda,\mu} = L_{c\lambda,c\mu}\) for nonzero \(c \in \mathbb{R}\).

(3) True/false (10 points each): If true, explain why. If false, give a counterexample:

(i) Suppose \(V = V_1 + V_2 + V_3\). Then, \(V = V_1 \oplus V_2 \oplus V_3\) if and only if \(0 = V_1 \cap V_2 = V_1 \cap V_3 = V_2 \cap V_3\).

False: one counterexample is \(V = \mathbb{R}^2\), with \(V_1 = \) the \(x\)-axis, \(V_2 = \) the \(y\)-axis, and \(V_3 = \) the line \(x = y\). Then, \(V = \mathbb{R}^2 = V_1 + V_2 + V_3\), but \(V_1 \cap V_2 = 0\) for all \(i \neq j\).

(ii) Suppose \(V = V_1 \oplus \cdots \oplus V_n\). Then a linear transformation \(T : V \to W\) is uniquely determined by its restrictions \(T|_{V_i} : V_i \to W\), which can be arbitrary linear maps (i.e., all possible collections \(\{V_i \to W\}\) of linear maps are obtainable in this way).

True: indeed, suppose that \(T\) and \(S\) have the same restrictions \(S|_{V_i} = T|_{V_i}\) for all \(i\). Then, \((T - S)|_{V_i} = 0\) for all \(i\). Now, for all \(v \in V\), write \(v = v_1 + \cdots + v_n\) for \(v_i \in V_i\). Then, \(T(v_1 + \cdots + v_n) = T(v_1) + \cdots + T(v_n) = 0\). Hence, the restrictions uniquely determine \(T\). To see that the restrictions can be arbitrary, let \(T_i : V_i \to W\) for all \(i\). Then, define \(T : V \to W\) as follows: for all \(v \in V\), write \(v\) uniquely as \(v = v_1 + \cdots + v_n\) for \(v_i \in V_i\). Then set \(T(v_1 + \cdots + v_n) = T(v_1) + \cdots + T(v_n)\). One can easily verify that the resulting \(T\) is linear (because \(\lambda v = \lambda v_1 + \cdots + \lambda v_n\), and if \(u = u_1 + \cdots + u_n\) for \(u \in V\), then \((u + v) = (u_1 + v_1) + \cdots + (u_n + v_n)\).

(iii) Let \(S, T \in L(V)\), where \(V\) is finite dimensional. Then, \(ST\) is invertible if and only if \(TS\) is invertible. **(Bonus 5 points (no credit for merely “true” or “false” answer): what about the case where \(V\) is infinite dimensional?)**

True: Suppose that \(ST\) is invertible. Then, \(\text{null}(ST) = 0\). In particular, \(\text{null}(T) = 0\), so \(T\) is injective, and hence invertible since \(V\) is finite dimensional. Also, \(\text{range}(S) = V\),
so $S$ is surjective and hence invertible, again because $V$ is finite dimensional. Then, $TS$ is invertible, with inverse $S^{-1}T^{-1}$.

Bonus: in the infinite-dimensional case, the statement is false. One counterexample is the forward and backward shift operators: if $T(a_0, a_1, a_2, \ldots) = (a_1, a_2, \ldots)$ is the backwards shift and $S(a_0, a_1, a_2, \ldots) = (0, a_0, a_1, \ldots)$ is the forwards shift, then $TS = I$ but $ST$ has a nonzero nullspace, $\{(a_0, 0, 0, \ldots) : a_0 \in \mathbb{F}\}$. Another example is the polynomials $P(\mathbb{F})$, where $T$ is differentiation and $S$ is indefinite integration “with $c = 0$,” i.e., $S(a_0 + a_1 x + \cdots + a_n x^n) = a_0 x + \frac{1}{2} a_1 x^2 + \cdots + \frac{1}{n+1} a_n x^{n+1}$. (Remark: an example can be found for every infinite-dimensional vector space, since every infinite-dimensional vector space can be written as $P(\mathbb{F}) \oplus U$ for some $U$, and so one can let $T = \frac{d}{dx} \oplus \text{Id} |_U$ and $S = \int \oplus \text{Id} |_U$, using the indefinite integration defined above. But you aren’t supposed to know this.)

(4) Computations (15 points each). **Show your work.** We will use the following matrix:

$$A := \begin{pmatrix} 1 & 2 & -3 & 4 \\ -2 & -4 & 6 & -6 \\ 1 & 0 & -3 & 2 \end{pmatrix}$$

(i) Compute the row echelon or reduced row echelon form matrix of $A$.

See (ii) for the computation. The result is

$$\begin{pmatrix} 1 & 2 & -3 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The reduced row echelon form matrix is then:

$$\begin{pmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(ii) Find all solutions of the equation $A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 5 \\ -4 \\ 3 \end{pmatrix}$.

We perform Gaussian elimination on $A$ and simultaneously on the vector $\begin{pmatrix} 5 \\ -4 \\ 3 \end{pmatrix}$:

$$\begin{pmatrix} 1 & 2 & -3 & 4 & | & 5 \\ -2 & -4 & 6 & -6 & | & -4 \\ 1 & 0 & -3 & 2 & | & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & -3 & 4 & | & 5 \\ 0 & 0 & 0 & 2 & | & 6 \\ 0 & -2 & 0 & -2 & | & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & -3 & 4 & | & 5 \\ 0 & -2 & 0 & -2 & | & -2 \\ 0 & 0 & 0 & 2 & | & 6 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & -3 & 4 & | & 5 \\ 0 & 1 & 0 & 1 & | & 1 \\ 0 & 0 & 0 & 1 & | & 3 \end{pmatrix}.$$
The free variable is \( x_3 \). By back substitution, we obtain the following space of solutions:

\[
\begin{pmatrix}
-3 + 3x_3 \\
-2 \\
x_3 \\
3
\end{pmatrix} : x_3 \in \mathbb{F}
\]

(0.8)

**Bonus 10 points:** Compute the \( A = PLU \) decomposition.

To do this, note that in doing Gaussian elimination, we applied one permutation, namely swapping the second and third rows. In order for \( L^{-1}P^{-1}A \) to be in row echelon form, \( P^{-1} \) is the permutation we apply, which swaps the second and third rows. So \( P \) is the inverse of this [which is actually the same matrix, since here \((P^{-1})^2 = I\)]:

\[
P = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}.
\]

(0.9)

Now, \( L^{-1} = E_m \cdots E_1 \) where left multiplication by \( E_i \) is the \( i \)-th row operation. Here, we have four row operations we need to perform on \( P^{-1}A \), corresponding to the following matrices \( E_1, E_2, E_3, \) and \( E_4 \), respectively:

\[
\begin{pmatrix}
1 & 0 & 0 \\
-1 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
2 & 0 & 1
\end{pmatrix}, \quad \begin{pmatrix}
1 & 0 & 0 \\
0 & -1/2 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1/2
\end{pmatrix}.
\]

(0.10)

Inverting these gives \( E_i^{-1} \). Therefore, \( L = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1} \) is

\[
L = \begin{pmatrix}
1 & 0 & 0 \\
1 & -2 & 0 \\
-2 & 0 & 2
\end{pmatrix}.
\]

(0.11)

Finally, row reduction on \( P^{-1}A \) yields the same row echelon form matrix \( U \) we found earlier (which also equals \( L^{-1}P^{-1}A \)):

\[
U = \begin{pmatrix}
1 & 2 & -3 & 4 \\
0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

(0.12)

Altogether, we obtain \( A = PLU \), i.e.,

\[
\begin{pmatrix}
1 & 2 & -3 & 4 \\
-2 & -4 & 6 & -6 \\
1 & 0 & -3 & 2
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 \\
1 & -2 & 0 \\
0 & 1 & 0
\end{pmatrix} \begin{pmatrix}
1 & 2 & -3 & 4 \\
0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

(0.13)

(5) Proofs (15 points each):

- Prove that \( U := \{ f \in P_n(\mathbb{F}) : f(1) = 0, f'(2) = 0 \} \) is a subspace of \( P_n \) and compute its dimension. Here, \( f' \) denotes the derivative of \( f \) (i.e., \((a_0 + \cdots + a_n x^n)' = a_1 + 2a_2 x + \cdots + na_n x^{n-1}\)).

Then, use this to conclude that \( \{ f \in P(\mathbb{F}) : f(1) = 0, f'(2) = 0 \} \) is infinite dimensional (here we dropped the subscript of \( n \)).

\( U \) is the nullspace of the map \( T : P_n(\mathbb{F}) \to \mathbb{F}^2 \) given by \( T(f) = (f(1), f'(2)) \). This map is linear because \( f(1) + g(1) = (f + g)(1) \) and \( \lambda f(1) = (\lambda f)(1) \), and similarly for derivatives (using either the sum and product rules for differentiation, or else the direct formula for derivative of a polynomial). This in particular shows that \( U \) is a subspace.
We claim that $T$ is surjective provided that $n \geq 1$. Indeed, $T(a_0 + a_1 x) = (a_0 + a_1, 2a_1)$, and this is clearly surjective $((x, y) = ((x - y/2) + (y/2), 2(y/2)))$. Therefore, $\dim U = \dim \mathcal{P}_n(F) - \dim F^2 = (n + 1) - 2 = n - 1$, provided $n \geq 1$. In the case $n = 0$, clearly the given subspace is $\{0\}$ which is zero-dimensional.

Finally, since $\{ f \in \mathcal{P}(F) : f(1) = 0, f'(2) = 0 \}$ contains each of the above subspaces, if it were finite dimensional, its dimension would be at least $n - 1$ for all $n \geq 1$. This is impossible, so $U$ must be infinite dimensional.

• Let $f(x)$ be a polynomial and $T \in \mathcal{L}(V)$. Prove that null $f(T)$ and range $f(T)$ are $T$-invariant subspaces of $V$ (recall that $U$ is $T$-invariant if $T(U) \subseteq U$). Show that this implies, in particular, that every eigenspace of $T$ is $T$-invariant.

First, we show that null $f(T)$ is $T$-invariant. Suppose $f(T)v = 0$. Then, $0 = Tf(T)v = f(T)Tv$, so $Tv \in \text{null}(f(T))$. Next, we show that range $f(T)$ is $T$-invariant. Again, suppose that $w = f(T)v$ is in the range of $f(T)$. Then, $T w = T f(T)v = f(T)Tv$, so $T w$ is also in the range.

For the final implication, take $f(T) = (T - \lambda I)$. Then, null $f(T)$ is the $\lambda$-eigenspace. We deduce from the above that it must be $T$-invariant.

• Let $T, S \in \mathcal{L}(V, W)$. Prove that the following are equivalent:

1. There exist bases $(v_1, \ldots, v_n)$, $(v'_1, \ldots, v'_n)$ of $V$ and $(w_1, \ldots, w_m)$, $(w'_1, \ldots, w'_m)$ such that $M_{(v_i), (w_i)}(T) = M_{(v'_i), (w'_i)}(S)$. That is, the matrix of $T$ in some bases equals the matrix of $S$ in some other bases.

2. $\text{rk}(T) = \text{rk}(S)$.

(Hint for (b) $\Rightarrow$ (a): pick nice bases for each of $T$ and $S$. Normal form!)

First, we show that (a) implies (b). We know that

$$\text{rk}(T) = \dim \text{range}(T) = \dim \text{colspan}(M_{(v_i), (w_i)}(T)) = \text{rk} M_{(v_i), (w_i)}(T).$$

By (a), $\text{rk} M_{(v_i), (w_i)}(T) = \text{rk} M_{(v'_i), (w'_i)}(S)$, and hence $\text{rk}(T) = \text{rk}(S)$.

Next, we show that (b) implies (a). Let $k = \text{rk}(T) = \text{rk}(S)$. Let $n := \dim V$, so then $\dim \text{null}(T) = n - k = \dim \text{null}(S)$. Pick bases $(v_{k+1}, \ldots, v_n)$ and $(v'_{k+1}, \ldots, v'_n)$ of $\text{null}(T)$ and $\text{null}(S)$, respectively, and extend these to bases $(v_1, \ldots, v_n)$ and $(v'_1, \ldots, v'_n)$ of $V$. Next, let $w_i := Tv_i$ and $w'_i := Sv'_i$ for $1 \leq i \leq k$. Then, $(w_1, \ldots, w_k)$ and $(w'_1, \ldots, w'_k)$ are bases of range $T$ and range $S$, since they clearly span (as $T(v_i)$ and $S(v'_i)$ span for all $i$, but for $i > k$ these are zero), and their length is equal to the rank of $T$ and $S$, i.e., the dimension of the range. So, they are linearly independent, and we can extend them to bases $(w_1, \ldots, w_n)$ and $(w'_1, \ldots, w'_n)$ of $W$. Then, in these bases, we have the block form matrices

\[
(0.14) \quad M_{(v_i), (w_i)}(T) = \begin{pmatrix}
I_k & 0_{k,n-k} \\
0_{m-k,0} & 0_{m-k,m-k}
\end{pmatrix} = M_{(v'_i), (w'_i)}(S).
\]

In other words, we can pick bases so that $T$ and $S$ are both written in normal form, and these normal forms are the same since they have the same rank. This proves the desired result.