PART I:

(1) Define the following (5 points each). Do not assume that any vector spaces are finite-dimensional unless specified.

(i) A direct sum $V = U \oplus W$ of vector subspaces, AND:
A complement of a vector subspace $U$.

Solution: A direct sum $V = U \oplus W$ is a pair of subspaces $U, W \subseteq V$ such that every vector $v \in V$ has a unique expression of the form $v = u + w$ for $u \in U$ and $w \in W$. A complement $W$ of $U$ is a subspace $W$ such that $V = U \oplus W$.

(ii) A matrix in Jordan canonical form

Solution: This is a block-diagonal matrix whose blocks are upper triangular with a single value $\lambda$ in all the diagonal entries, and 1’s immediately above the diagonal, and zeros everywhere else.

(iii) The characteristic polynomial of a linear transformation $T \in \mathcal{L}(V)$, for $V$ a finite-dimensional complex vector space using eigenvalues and generalized eigenspaces.

Solution: This is $\prod \lambda(x - \lambda)^{\dim V(\lambda)}$ where the product is over all the eigenvalues $\lambda$ of $T$, and $V(\lambda)$ is the generalized eigenspace of eigenvalue $\lambda$.

(iv) The sign of a permutation $\sigma \in S_n$ (you cannot assume the definition of the function $o(\sigma)$ is known)

Solution: This is $(-1)^{o(\sigma)}$ where $o(\sigma) = |\{(i, j) : 1 \leq i, j \leq n, i < j, \sigma(i) > \sigma(j)\}|$.

(v) The determinant of a matrix over an arbitrary field using the sum formula.

Bonus 5 points: Give the definition using a uniqueness property—but don’t spend too much time on this before you finish the rest of the test.

Solution: Let $A = (a_{ij})$ be a square matrix of size $n \times n$. Then $\det(A) = \sum_{\sigma \in S_n} \text{sign}(\sigma)a_{\sigma(1),1} \cdots a_{\sigma(n),n}$.

Bonus: This is the unique function on matrices that satisfies multilinearity, alternation, and such that $\det(I) = 1$. Here multilinearity means that the function is linear in each column separately, i.e., it is multiadditive and multihomogeneous: multiadditive means that, if $A$ and $B$ are two matrices which have the same columns except for the $k$-th column, and $C$ is the matrix which has the same columns except for the $k$-th column which is the sum of the $k$-th columns of $A$ and $B$, then $\det(C) = \det(A) + \det(B)$. Similarly, multihomogeneous means that if $A$ and $B$ are two matrices which are the same except in the $k$-th column where the $k$-th column of $B$ is $\lambda$ times the $k$-th column of $A$, then $\det(B) = \lambda \det(A)$. Finally, alternation means that, if $A$ is a matrix with two identical columns, then $\det(A) = 0$. 
(vi) The characteristic polynomial of $T \in \mathcal{L}(V)$ for $V$ a finite-dimensional vector space over an arbitrary field $F$ using matrices, but you cannot assume the definition of the characteristic polynomial of a matrix.

Solution: This is $\chi_T(x) := \det(xI - A)$.

(2) Problems (10 points each):

(i) Describe all possible vector spaces $V$ and operators $T \in \mathcal{L}(F^4, V)$ such that $\text{null} \, T = \{(x_1, x_2, x_3, x_4) \in F^4 : x_1 = 4x_2$ and $x_3 = 5x_4\}$. What can the dimension of $V$ be (and need it be finite-dimensional)?

Solution: The answer is that $V$ can be any vector space of dimension $\geq 2$ (either finite-dimensional of this dimension, or infinite-dimensional), and then the set of all possible $T$ is the same as the set of linearly independent lists $(u, v)$ of length two in $V$, under the correspondence $T(x_1, x_2, x_3, x_4) = (x_1 - 4x_2)u + (x_3 - 5x_4)v$. The reason this works is that we can write the direct sum decomposition

$$F^4 = \{(x_1, 0, x_3, 0) : x_1, x_3 \in F\} \oplus \text{null} \, T = \{(4x_2, x_2, 5x_4, x_4) : x_2, x_4 \in F\},$$

and then operators $T$ with the above nullspace are uniquely determined by their restriction to the complement $\{(x_1, 0, x_3, 0) : x_1, x_3 \in F\}$, which is in turn uniquely determined by the vectors $u = T(1, 0, 0, 0)$ and $v = T(0, 0, 1, 0)$, since $(1, 0, 0, 0)$ and $(0, 0, 1, 0)$ form a basis for this complement. Given these $u$ and $v$, the operator $T$ then has the formula given above.

(ii) Suppose that $S, T \in \mathcal{L}(V)$, where $V$ is a finite-dimensional vector space over an arbitrary field $F$. Show that

(a) $\text{rk}(S + T) \leq \text{rk}(S) + \text{rk}(T)$;

(b) Suppose $\text{rk}(S + T) = \dim V = \text{rk}(S) + \text{rk}(T)$. Show that $V = \text{null}(S) \oplus \text{null}(T)$.

Solution: (a) Note that $\text{range}(S + T) \subseteq \text{range}(S) + \text{range}(T)$, and $\dim(\text{range}(S) + \text{range}(T)) \leq \dim \text{range}(S) + \dim \text{range}(T) = \text{rk} \, S + \text{rk} \, T$, since a spanning list for range $S$ and range $T$ can be combined to yield a spanning list for the sum range $S + \text{range} \, T$. Hence $\text{rk}(S + T) = \dim \text{range}(S + T) \leq \text{rk} \, S + \text{rk} \, T$.

(b) Under the assumptions given, the rank-nullity theorem shows that $\dim \text{null} \, S = \dim V - \text{rk} \, S = \text{rk} \, T$ and $\dim \text{null} \, T = \dim V - \text{rk} \, T = \text{rk} \, S$. So also $\dim \text{null} \, S + \text{null} \, T = \dim V$. Therefore we only have to prove that $\text{null} \, S \cap \text{null} \, T = \{0\}$. This follows because $\text{null} \, S \cap \text{null} \, T \subseteq \text{null} \, (S + T)$, which is zero by the rank-nullity theorem, since $\text{rk}(S + T) = \dim V$ (the latter implies that $\dim \text{null} \, S \cap \text{null} \, T = 0$ so that $\text{null} \, S \cap \text{null} \, T = 0$).

(iii) Let $F = \mathbb{R}$ or $\mathbb{C}$, and consider $F^2$ as an inner product space equipped with the dot product. Suppose $T \in \mathcal{L}(F^2)$ is a normal operator such that $T(a, a) = 0$ and $T^2 = 3T$. Find all possible $T$, in terms of a formula for $T(w)$ for arbitrary $w$. How many such $T$ are there?

Solution: Since $T$ is normal, the spectral theorem implies that it admits an orthonormal eigenbasis. Hence, the orthogonal complement of the invariant subspace $\text{Span}((1, 1))$ must also be invariant, and therefore the vector $(1, -1)$ must be an eigenvector, say with eigenvalue $\lambda$. Then $T$ is uniquely determined by this eigenvalue $\lambda$, and it has the form $T(a, b) = \frac{1}{2}(a-b, b-a)$. (Alternatively, it has the form $T(w) = \lambda \langle v, w \rangle v$, where $v = \frac{1}{\sqrt{2}}(1, -1)$ is an eigenvector of eigenvalue $\lambda$ of norm one, which extends the norm one eigenvector $\frac{1}{\sqrt{2}}(1, 1)$ to an orthonormal eigenbasis of $T$.)
So it remains to see what the possible values of λ are. Since $T^2 = 3T$, we deduce that $\lambda^2 v = T^2 v = 3Tv = 3\lambda v$ for a nonzero eigenvector $v$ of eigenvalue $\lambda$, and then this implies that $(\lambda^2 - 3\lambda)v = 0$, i.e., $\lambda = 0$ or $3$. Conversely, either $\lambda = 0$ or $3$ yields an operator $T$ such that $T^2 v = 3Tv$ and since also $T^2(1,1) = 0 = 3T(1,1)$, the resulting $T$ satisfies $T^2 = 3T$.

(3) True/false (10 points each): If true, explain why. If false, give a counterexample. Suggestion: Write a sketchy answer at first and come back later if you have time!

(i) Let $V$ be a finite-dimensional complex vector space of dimension $n > 1$. Then, for every $T \in \mathcal{L}(V)$, all generalized eigenvectors of $T$ are eigenvectors of $T^n$.

Solution: False. This is actually false whenever $T$ has a Jordan block of size $> 1$ with a nonzero eigenvalue. Take the case where $\mathcal{M}(T)$ is a single Jordan block of size $n > 1$ with eigenvalue $\lambda \neq 0$. Then $\mathcal{M}(T^n)$ has $\lambda^n$ on the diagonal and $n\lambda^{n-1} \neq 0$ in every entry directly above the diagonal. Thus, $T^n$ has only one eigenvalue, $\lambda^n$, and the eigenspace is also only one-dimensional, since $\dim\ker(\mathcal{M}(T^n) - \lambda^n I) = 1$ for such a matrix. Since we assumed $n > 1$, it follows that $V$ contains (nonzero) vectors that are not eigenvectors of $T^n$. On the other hand, in this case, $V = V(\lambda)$, i.e., every vector is a generalized eigenvector. So not all generalized eigenvectors of $T$ are eigenvectors of $T^n$.

(ii) Suppose $(e_1, \ldots, e_n)$ and $(e'_1, \ldots, e'_n)$ are orthonormal lists in an inner product space such that $\text{Span}(e_1, \ldots, e_k) = \text{Span}(e'_1, \ldots, e'_k)$ for all $1 \leq k \leq n$. Then, for all $k$, $e'_k = \lambda_k e_k$ for some $\lambda_k$ such that $|\lambda_k| = 1$.

Solution: True. First of all since $\text{Span}(e_1) = \text{Span}(e'_1)$, it follows that $e'_1 = \lambda_1 e_1$ for some $\lambda$. Next, note that, in general, if $e'_k = \lambda_k e_k$ for some $\lambda_k$, then taking norm we get $1 = |e'_k| = |\lambda_k e_k| = |\lambda_k|$. So we only have to prove inductively that there exists $\lambda_k$ such that $e'_k = \lambda_k e_k$ for $k \geq 2$.

Lemma 0.1. For all $2 \leq k \leq n$, $\text{Span}(e_k) = \text{Span}(e_1, \ldots, e_{k-1})^\perp \cap \text{Span}(e_1, \ldots, e_k)$.

Proof. It is clear from orthonormality that the inclusion $\subseteq$ holds. Next, restricted to $\text{Span}(e_1, \ldots, e_k)$, $\dim \text{Span}(e_1, \ldots, e_{k-1})^\perp = k - (k - 1) = 1$. So both sides have the same dimension, and we conclude the equality. □

The lemma also holds, with the same proof, replacing all $e_j$ by $e'_j$. Hence, applying the assumption in the statement of the problem, $\text{Span}(e_k) = \text{Span}(e_1, \ldots, e_{k-1})^\perp \cap \text{Span}(e_1, \ldots, e_j) = \text{Span}(e_1', \ldots, e_{k-1}')^\perp \cap \text{Span}(e_1', \ldots, e_j') = \text{Span}(e_k')$. So there exists $\lambda_k$ such that $e'_k = \lambda_k e_k$, as desired.

(iii) Suppose $T \in \mathcal{L}(\mathbb{R}^2)$ is an operator (with $\mathbf{F} = \mathbb{R}$) with no eigenvalues. Then, in some orthonormal basis, $\mathcal{M}(T) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. (Hint: does there exist a non-normal operator with no eigenvalues?)

Solution: False. In fact, as we saw in class, in an orthonormal basis, any normal operator on $\mathbb{R}^2$ with no eigenvalues must be of the given form, so we could pick any matrix with irreducible characteristic polynomial not of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ and it would not be normal. In particular we can take, e.g., $\begin{pmatrix} 0 & -(a^2 + b^2) \\ 1 & 2a \end{pmatrix}$. 3
More directly, we can use only that any $T$ with $M(T) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ in an orthonormal basis must be normal (the real spectral theorem, or by direct computation); and then we can directly see that $A = \begin{pmatrix} 0 & -(a^2 + b^2) \\ 1 & 2a \end{pmatrix}$ has no eigenvalues and is not normal.

Indeed, $A^*A = \begin{pmatrix} 1 & 2a \\ 2a & (2a)^2 + (a^2 + b^2)^2 \end{pmatrix}$ (the matrix of dot products of columns of $A$) whereas $AA^* = \begin{pmatrix} (a^2 + b^2)^2 & -2a(a^2 + b^2) \\ -2a(a^2 + b^2) & 1 + (2a)^2 \end{pmatrix}$ (the matrix of dot products of rows of $A$).

**Part II:**

(4) **Computations (15 points each).** Show your work.

(i) Compute the trace (5 points) and determinant (10 points) of the following matrix. For determinant, make sure to use row and/or column operations (remembering how this affects determinant), and not merely the sum formula!

\[
\begin{pmatrix}
-44 & 18 & -66 \\
-30 & 10 & -48 \\
24 & -9 & 37
\end{pmatrix}
\]

**Bonus up to 10 points:** Find the eigenvalues of the above matrix using its trace and determinant. Hint: look for integer eigenvalues, then prove they are the eigenvalues.

**Solution:** The trace is easy: $-44 + 10 + 37 = 3$. The determinant can be computed easily with a couple row and column operations, e.g.:

\[
\begin{pmatrix}
-44 & 18 & -66 \\
-30 & 10 & -48 \\
24 & -9 & 37
\end{pmatrix} \mapsto \begin{pmatrix} 4 & 0 & 8 \\ -30 & 10 & -48 \\ 24 & -9 & 37 \end{pmatrix} \mapsto \begin{pmatrix} 4 & 0 & 0 \\ -30 & 10 & 12 \\ 24 & -9 & -11 \end{pmatrix},
\]

where first we added twice the third row to the first row, and then we subtracted twice the first column from the third column. Now, the resulting matrix has all zeroes in the first row except for the first entry, so by the sum formula for determinant, it equals 4 times the determinant of the $2 \times 2$-matrix in the lower right corner, which is $10 \cdot (-11) - 12 \cdot (-9) = -110 + 108 = -2$. So we get $-8$ for the determinant.

**Bonus:** If we look for all integer eigenvalues, they must sum to 3 and multiply to $-8$. All eigenvalues must be in the set $\{\pm 1, \pm 2, \pm 4, \pm 8\}$, since these are the factors of $-8$; the only possibility is $1, -2$, and $4$ (indeed, exactly one of the eigenvalues must be $\pm 1$ in order to get an odd sum, and then the other two are $\pm 2$ and $\pm 4$, and the must be exactly one minus sign to have product $-8$ and a positive sum, and the only way the sum is 3 is as mentioned). So the answer is: $1, -2, 4$, provided that the eigenvalues are integers.

To prove that $1, -2, 4$ are really the eigenvalues, it is enough to verify that just one of these is an eigenvalue, because then the sum and product of the remaining two would be determined from this one, and we know that the sum and product of two numbers determines their value (this is equivalent to saying that the characteristic polynomial gives the eigenvalues in the $2 \times 2$ case). So we pick $4$: then if $A$ denotes
our matrix,

\[
A - 4I = \begin{pmatrix}
-48 & 18 & -66 \\
-30 & 6 & -48 \\
24 & -9 & 33
\end{pmatrix},
\]

which is noninvertible since the first row is \(-2\) times the third row. So \(4\) is an eigenvalue.

(ii) Let \(T : \mathbb{R}^3 \to \mathbb{R}^3\) be the operator \(T(x, y, z) = (z, x, y)\).

(a) Verify that \(v_1 = (1, 1, 1)\) is a fixed vector, i.e., \(Tv_1 = v_1\), and find a two-dimensional invariant subspace \(U \subseteq \mathbb{R}^3\) (Hint: recall that isometries are normal on finite-dimensional spaces).

(b) (10 points:) Pick an orthonormal (real) basis \(v_2, v_3\) of \(U\) and write \(\mathcal{M}(T|_U)\) in this basis. (Note: it might save you time and/or help you to observe what property this matrix must have, using that \(T\) is an isometry; this is not necessary, however.)

Bonus 5 points: Find a (real) square root of \(T\) in terms of \(v_1, v_2,\) and \(v_3\) (and/or anything else you may need to say to describe it).

For your convenience I recall the formulas: \(\cos 0^\circ = 1 = \sin 90^\circ, \cos 30^\circ = \frac{\sqrt{3}}{2} = \sin 60^\circ, \cos 45^\circ = \frac{\sqrt{2}}{2} = \sin 45^\circ, \cos 60^\circ = \frac{1}{2} = \sin 30^\circ,\) and similarly \(\cos(-x) = \cos(x), \sin(-x) = -\sin(x),\) and \(\cos(x + 180^\circ) = -\cos(x)\) and \(\sin(x + 180^\circ) = -\sin(x).\)

Bonus another 5 points: Find all the real square roots of \(T\). How many are there?

Bonus yet another 5 points: What about the complex square roots of a matrix \(\mathcal{M}(T)\) in any basis: are there more of them, how many, and why?

Solution: (a) It follows immediately that \(T(1, 1, 1) = (1, 1, 1),\) there is nothing else one can say. Note that \(T\) is an isometry since it sends an orthonormal basis (the standard basis) to another orthonormal basis (a permutation of this basis, which is obviously still orthonormal). Hence it must send \(\text{Span}((1, 1, 1))^\perp\) to \(\text{Span}(T(1, 1, 1))^\perp\). So this is an invariant two-dimensional space \(U\). (Alternatively, since \(T\) is an isometry as above, it is normal, and we know that a normal transformation admits an orthonormal basis in which it is block diagonal with one-by-one and two-by-two blocks, where the one-by-one blocks of a particular value \(\lambda\) correspond to the \(\lambda\)-eigenspace. So in particular the perpendicular of any eigenspace is an eigenspace. So in particular the perpendicular of any eigenspace is invariant, i.e., \(\text{Span}(1, 1, 1)^\perp\) is invariant.)

This orthogonal complement is \(U := \{(x, y, z) : x + y + z = 0\}.\) It is indeed an invariant subspace of dimension two.

(b) We can take \(v_2 = \frac{1}{\sqrt{2}}(1, -1, 0)\) and \(v_3 = \frac{1}{\sqrt{6}}(1, 1, -2).\) Clearly \(v_2 \cdot v_2 = 1 = v_3 \cdot v_3,\) and these are in \(U,\) and orthogonal to each other. Now, \(T(v_1) = \frac{1}{\sqrt{2}}(0, 1, -1)\) which is \(\frac{\sqrt{2}}{2}v_3\) plus some multiple of \(v_2\) looking at the third entry; looking at the first entry that multiple must be \((-\sqrt{2})(\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{6}}) = -\frac{1}{2}.)\) And indeed, \(T(v_1) = -\frac{1}{2}v_1 + \frac{\sqrt{3}}{2}v_2.\) We can then compute \(T(v_2):\) up to \(\pm 1\) this is actually determined as the unique vector completing \(T(v_1)\) to the orthonormal basis \((T(v_1), T(v_2))\) of \(U\) (note that \(T\) is an isometry). So we get \(T(v_2) = \pm (\frac{\sqrt{3}}{2}v_1 + \frac{1}{2}v_2).\) The sign here is negative by looking at the third coefficient: note that \(T(v_2) = \frac{1}{\sqrt{6}}(-2, 1, 1).\) Thus the matrix we get is:

\[
\mathcal{M}(T|_U) = \begin{pmatrix}
-\frac{1}{2} & -\frac{\sqrt{3}}{2} \\
\frac{\sqrt{3}}{2} & -\frac{1}{2}
\end{pmatrix}.
\]
**Bonus:** We can rewrite $T|_U$ as
\[
\begin{pmatrix}
\cos 120^\circ & -\sin 120^\circ \\
\sin 120^\circ & \cos 120^\circ
\end{pmatrix},
\]
a rotation matrix of degree $120^\circ$. So one square root of $T|_U$ is the rotation by $60^\circ$; another would be rotation by $(180 + 60)^\circ = 240^\circ$. (In fact these are the only possible square roots, as we explain below.) Putting this together, we get square roots of the form $S(v_1) = \pm v_1$, and $S|_U = $ the rotation matrix of angle either $60^\circ$ or $240^\circ$. These are four square roots.

We claim that these are all possible real square roots. Any square root must have complex eigenvalues which are square roots of $-\frac{1}{2} \pm \frac{\sqrt{3}}{2} = \cos 120^\circ \pm \sin 120^\circ i = e^{\pm 2\pi i/3}$, and thus the only possible square roots are $e^{\pm 2\pi i/6}$ with possibly different values $\pm$, i.e., $\pm \cos 60^\circ \pm \sin 60^\circ = \pm \frac{1}{2} \pm \frac{\sqrt{3}}{2}$ with possibly different values $\pm$ of the two signs appearing. But in a real two-by-two matrix, the two complex eigenvalues are complex conjugates of each other, so there are only two possible pairs of complex eigenvalues. Since both $T|_U$ and its square root have one-dimensional eigenspaces which therefore must be identical, the eigenvalues uniquely determine the square root, so $T|_U$ admits only two different square roots: the rotation matrices by angles $60^\circ$ and $240^\circ$. Finally, any square root $S$ of $T$ is uniquely determined by the eigenvalue of $v_1$ (which must be an eigenvector of both since $S$ and $T$ both have one-dimensional eigenspaces which therefore are the same), which is $\pm 1$, together with $S|_U$, which is as described.

Finally, we saw in the above analysis that $\mathcal{M}(T|_U)$, as a complex matrix, actually admits four square roots: we can pick separately the square roots of the eigenvalues of the two orthogonal one-dimensional eigenspaces of $T$ on $U$. Putting this together with the choice $\pm$ of the eigenvalue of $v_1$, $\mathcal{M}(T)$ actually admits eight square roots as a complex matrix, as opposed to the four mentioned above as a real matrix. This is because a real square root must have eigenvalues closed under taking complex conjugation, and only half the complex square roots have this property.

(iii) Compute a Jordan basis for the linear transformation $T \in \mathcal{L}(\text{Mat}(3, 1, \mathbb{R}))$ given by $T(v) = Av$, where
\[
A = \begin{pmatrix}
-13 & -1 & 4 \\
13 & 1 & -4 \\
-39 & -3 & 12
\end{pmatrix}.
\]

Then, write the Jordan form matrix $\mathcal{M}(T)$. **Hint:** First find the nullspace and trace. **Your answer should consist of two things:** a Jordan basis, and the Jordan matrix $\mathcal{M}(T)$ when $T$ is written in this basis. Take care with the (ordering of) the basis so that it is true that the matrix is the one you gave!

**Solution:** Note that this matrix has rank one since it is nonzero and all rows (equivalently, all columns) are multiples of each other. So, its nullspace has dimension two, which is easy to write down (it is $\text{Span}\left(\begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix}, (1 \ -1 \ 3)\right)$). Furthermore, its trace is zero. So the eigenvalues with multiplicity (i.e., the values appearing on the diagonal for any upper-triangular matrix of the transformation) must include two zero values and sum to zero. Thus all three eigenvalues are all zero. Therefore, $V = V(0)$ and the matrix $A$ is nilpotent.

To compute a Jordan basis, we need only pick a vector not in the nullspace, $v_1$, add in $T(v_1)$, and then throw in one more vector in the nullspace which is not a multiple
of $T(v_1)$. For example, we can take $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and then $Tv_1 = \begin{pmatrix} -13 \\ 13 \\ -39 \end{pmatrix}$. Then we can take $v_2 = \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix}$ as we saw this is in the nullspace and is not a multiple of $Tv_1$. Then the basis $(v_2, Tv_1, v_1)$ is a Jordan basis, and in this basis, the matrix is

$$M(T) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

(5) Proofs (15 points each):

(i) Let $T \in \mathcal{L}(\mathbb{C}^n)$ be the operator $T(z_1, \ldots, z_n) = (z_n, z_1, z_2, \ldots, z_{n-1})$. Find all (complex) eigenvalues and eigenvectors of $T$.

**Bonus 5 points:** Using your answer to (a), find the determinant $\det(T)$.

**Bonus 5 more points:** By definition of sign, this is also the sign of the permutation $\sigma$ such that $\sigma(j) = j + 1$ for $1 \leq j \leq n - 1$ and $\sigma(n) = 1$. So deduce a formula for $\text{sign}(\sigma)$. Show that this matches the answer you get using the definition of sign directly.

**Remark.** All permutations $\sigma \in S_n$ decompose as a disjoint union of permutations $i_1 \mapsto i_2 \mapsto \cdots \mapsto i_m \mapsto i_1$ as above, simply by iterating $\sigma$. Then the sign of $\sigma$ is the product of the signs computed above over all $m$. Thus this is computable in $O(n)$ time (whereas the definition of sign using order could be $O(n^2)$).

**Solution:** An eigenvector of eigenvalue $\lambda$ must satisfy $T(z_1, \ldots, z_n) = (z_n, z_1, z_2, \ldots, z_{n-1}) = \lambda (z_1, \ldots, z_n)$. Thus $z_2 = \lambda z_1, z_3 = \lambda z_2, \ldots, z_n = \lambda z_{n-1},$ and $z_1 = \lambda z_n$. Substituting $n$ times, we obtain that $z_1 = \lambda^n z_1$. Then either $\lambda = 0$, in which case $(z_1, \ldots, z_n)$ is zero, or else $\lambda^n = 1$, so $\lambda$ is an $n$-th root of unity. Moreover, all $n$-th roots of unity are eigenvalues: if $\lambda^n = 1$, then we can always set $(z_1, \ldots, z_n) = (1, \lambda, \ldots, \lambda^{n-1})$. So there are $n$ distinct eigenvalues, which are the $n$-th roots of unity, $e^{2k\pi i/n}$ for $k = 0, 1, \ldots, n - 1$. This means that the operator $T$ admits an eigenbasis, given by the collection of $(1, \lambda, \ldots, \lambda^{n-1})$ as $\lambda$ ranges over all the eigenvalues described. In other words, for each eigenvalue $\lambda = e^{2k\pi i/n}$, the eigenvectors of $\lambda$ are exactly $(z_1, e^{2k\pi i/n}z_1, \ldots, e^{2k(n-1)\pi i/n}z_1)$.

**Bonus:** The determinant is the product of the eigenvalues, so we get

$$\det(T) = \prod_{k=0}^{n-1} e^{2k\pi i/n} = e^{2(0+1+\cdots+(n-1))\pi i/n} = e^{n(n-1)\pi i/n} = e^{(n-1)\pi i/n} = (-1)^{n-1}.$$  

So $\det(T) = (-1)^{n-1}$.

This is indeed the sign of $\sigma$: $o(\sigma) = \{(1, j) : 2 \leq j \leq n\} = n - 1$, so that $\text{sign}(\sigma) = (-1)^{o(\sigma)} = (-1)^{n-1}$.

(ii) Let $T$ be a normal operator on a finite-dimensional complex inner product space $V$.

(a) Prove that there exists a square root $S$ such that $S^2 = T$.

(b) Now suppose further that $T$ is invertible (as well as normal). Prove that all square roots $S$ (i.e., operators $S$ such that $S^2 = T$) are normal if and only if $T$ has $n = \dim V$ distinct eigenvalues.
(a) Pick any orthonormal eigenbasis of \(T\) (which exists by the complex spectral theorem), and let \(S\) be an operator with the same orthonormal eigenbasis and with eigenvalues any choice of square roots of the eigenvalues of \(T\). This is evidently a square root.

(b) If \(T\) has \(n\) distinct eigenvalues, then since these are also the squares of the eigenvalues of \(S\), \(S\) must also have \(n\) distinct eigenvalues. Therefore the one-dimensional eigenspaces of \(S\) must equal the one-dimensional eigenspaces of \(T\), and so \(T\) admits an orthonormal eigenbasis if and only if \(S\) does, i.e., \(T\) is normal if and only if \(S\) is.

For the converse, suppose that \(T\) does not have \(n\) distinct eigenvalues. Then there is some eigenvalue \(\lambda\) such that \(\dim \text{null}(T - \lambda I) \geq 2\). Since we assumed that \(T\) is invertible, \(\lambda \neq 0\). Now, it is enough to show that \(T|_{\text{null}(T - \lambda I)} = \lambda I|_{\text{null}(T - \lambda I)}\) has nonnormal square roots, since we can construct a square root of \(T\) by taking this on \(\text{null}(T - \lambda I)\) and any choice of square root on \(\text{null}(T - \lambda I)\) since \(T\) is normal. The result will not be normal if it is not normal on \(\text{null}(T - \lambda I)\).

Moreover, we can restrict to a 2-dimensional subspace \(U \subseteq \text{null}(T - \lambda I)\), and since \(\text{null}(T - \lambda I) = U \oplus U^\perp\) (with \(\perp\) taken inside \(\text{null}(T - \lambda I)\), it is enough to construct a nonnormal square root on \(U\). In other words we can restrict to the case \(n = 2\) and \(T = \lambda I\).

So let \(n = 2\) and \(T = \lambda I\). Pick a basis \((v_1, v_2)\) of \(V\) such that \(\langle v_1, v_2 \rangle \neq 0\). Let \(\mu\) be any square root of \(\lambda\). Then consider the square root \(S\) of \(T\) such that \(S(v_1) = \mu v_1\) and \(S(v_2) = -\mu v_2\). Clearly \(S^2 = T\). But \(S\) admits an eigenbasis with distinct eigenvalues, and this eigenbasis cannot be orthonormal since the two basis vectors are not orthogonal; all eigenbases are obtained from this one by rescaling the two basis vectors since the eigenvalues are distinct. So \(S\) does not admit an orthonormal eigenbasis, and hence it is not normal.