Understanding Resonant Systems

• Option 1: *Simulate the whole thing exactly*
  — many powerful numerical tools
  — limited insight into a single system
  — can be difficult, especially for weak effects (nonlinearities, etc.)

• Option 2: Solve *each component separately*, couple with explicit perturbative method (one kind of “coupled-mode” theory)

• Option 3: *abstract the geometry* into its most generic form
  …write down the *most general possible equations*
  …*constrain* by fundamental laws (conservation of energy)
  …solve for *universal behaviors* of a whole class of devices
  …characterized via specific *parameters from option 2*
“Temporal coupled-mode theory”

• Generic form developed by Haus, Louisell, & others in 1960s & early 1970s, many variations…

• Equations are generic ⇒ reappear in many forms in many systems, rederived in many ways (e.g. Breit–Wigner scattering theory)
  – full generality is not always apparent

(modern name coined by S. Fan @ Stanford)
TCMT example: a linear filter

= abstractly:
  two single-mode i/o ports + one resonance

resonant cavity
  frequency $\omega_0$, lifetime $\tau$
Temporal Coupled-Mode Theory
for a linear filter

\[
\frac{da}{dt} = -i \omega_0 a - \frac{2}{\tau} a + \sqrt{\frac{2}{\tau}} s_{1+}
\]

\[
s_{1-} = -s_{1+} + \sqrt{\frac{2}{\tau}} a, \quad s_{2-} = \sqrt{\frac{2}{\tau}} a
\]

assumes only:

• exponential decay
  (strong confinement)

• linearity

• conservation of energy

• time-reversal symmetry

\[|s|^2 = \text{power}\]
\[|a|^2 = \text{energy}\]
Temporal Coupled-Mode Theory
for a linear filter

Resonant cavity
frequency $\omega_0$, lifetime $\tau$

$|s|^2 = \text{flux}$

$|a|^2 = \text{energy}$

Transmission $T$

$$T = \text{Lorentzian filter}$$

$$T = \frac{4}{\tau^2} \frac{1}{(\omega - \omega_0)^2 + \frac{4}{\tau^2}}$$

Input $s_{1+}$ $s_{1-}$ $s_{2-}$ Output $s_{2-}$ $s_{1+}$ $a$

Transmission $T$

$$T = \frac{|s_{2-}|^2}{|s_{1+}|^2}$$
Resonant Filter Example

Lorentzian peak, as predicted.

An apparent *miracle*:

~ 100% transmission at the resonant frequency

cavity decays to input/output with *equal rates*

– At resonance, *reflected wave* destructively interferes with *backwards-decay from cavity* & the two *exactly cancel*. 
Some interesting resonant transmission processes

Wireless resonant power transfer
[ M. Soljacic, MIT (2007) ]
witricity.com

Resonant LED emission
luminus.com

(narrow-band) resonant absorption in a thin-film photovoltaic
[ e.g. Ghebrebrhan (2009) ]
Another interesting example: **Channel-Drop Filters**

Perfect channel-dropping if:

Two resonant modes with:
- even and odd symmetry
- equal frequency (degenerate)
- equal decay rates

Dimensionless Losses: $Q$

$Q = \omega_0 \tau / 2$

quality factor $Q = \#$ optical periods for energy to decay by $\exp(-2\pi)$

energy $\sim \exp(-\omega_0 t/Q) = \exp(-2t/\tau)$

in frequency domain: $1/Q = \text{bandwidth}$

from temporal coupled-mode theory:

$T = \text{Lorentzian filter}$

$\frac{1}{Q} = \frac{2}{\omega_0 \tau}$

$= \frac{4}{\tau^2} \frac{1}{(\omega - \omega_0)^2 + \frac{4}{\tau^2}}$

...quality factor $Q$
More than one $Q$…

A simple model device (filters, bends, …):

$\frac{1}{Q} = \frac{1}{Q_r} + \frac{1}{Q_w}$

$Q = \text{lifetime} / \text{period} = \text{frequency} / \text{bandwidth}$

We want: $Q_r \gg Q_w$

TCMT $\Rightarrow$

$1 - \text{transmission} \sim \frac{2Q}{Q_r}$

worst case: high-Q (narrow-band) cavities
Nonlinearities + Microcavities?

- weak effects
  \[ \Delta n < 1\% \]
- very intense fields
  & sensitive to small changes

A simple idea:
for the same input power, nonlinear effects are stronger in a microcavity

That's not all!

nonlinearities + microcavities = qualitatively new phenomena
Nonlinear Optics

Kerr nonlinearities $\chi^{(3)}$: \( \text{(polarization } \sim E^3) \)

- Self-Phase Modulation (SPM)
  \[ = \text{change in refractive index}(\omega) \sim |E(\omega)|^2 \]
- Cross-Phase Modulation (XPM)
  \[ = \text{change in refractive index}(\omega) \sim |E(\omega_2)|^2 \]

- Third-Harmonic Generation (THG) & down-conversion (FWM)
  \[ = \omega \to 3\omega, \text{ and back} \]
- etc…

Second-order nonlinearities $\chi^{(2)}$: \( \text{(polarization } \sim E^2) \)

- Second-Harmonic Generation (SHG) & down-conversion
  \[ = \omega \to 2\omega, \text{ and back} \]
- Difference-Frequency Generation (DFG) \[ = \omega_1, \omega_2 \to \omega_1-\omega_2 \]
- etc…
Nonlinearities + Microcavities?

weak effects
\( \Delta n < 1\% \)

very intense fields
& sensitive to small changes

A simple idea:
for the same input power, nonlinear effects
are stronger in a microcavity

That’s not all!
nonlinearities + microcavities
= qualitatively new phenomena

let’s start with a well-known example from 1970’s…
A Simple Linear Filter

Linear response:
Lorenzian Transmission
Filter + Kerr Nonlinearity?

Linear response: Lorenzian Transmission

Kerr nonlinearity: $\Delta n \sim |E|^2$

shifted peak?

+ nonlinear index shift $= \omega$ shift
Optical Bistability


Logic gates, switching, rectifiers, amplifiers, isolators, …

Power threshold $\sim V/Q^2$
(in cavity with $V \sim (\lambda/2)^3$, for Si and telecom bandwidth power $\sim$ mW)

Bistable (hysteresis) response
(& even multistable for multimode cavity)

[ Soljacic et al., PRE Rapid. Comm. 66, 055601 (2002). ]
TCMT for Bistability

[ Soljacic et al., PRE Rapid. Comm. 66, 055601 (2002). ]

\[ \frac{da}{dt} = -i(\omega_0 - \alpha |a|^2)a - \frac{2}{\tau}a + \sqrt{\frac{2}{\tau}}s_{1+} \]

\[ s_{1-} = -s_{1+} + \sqrt{\frac{2}{\tau}}a, \quad s_{2-} = \sqrt{\frac{2}{\tau}}a \]

resonant cavity frequency $\omega_0$, lifetime $\tau$,
SPM coefficient $\alpha \sim \chi^{(3)}$
(from perturbation theory)

\[ |s|^2 = \text{power} \]
\[ |a|^2 = \text{energy} \]

gives cubic equation for transmission
... bistable curve
Accuracy of Coupled-Mode Theory

[ Soljacic et al., PRE Rapid. Comm. 66, 055601 (2002). ]
Optical Bistability in Practice

Q ~ 30,000
V ~ 10 optimum
Power threshold ~ 40 µW

Q ~ 10,000
V ~ 300 optimum
Power threshold ~ 10 mW

[ Notomi et al. (2005). ]

[ Xu & Lipson, 2005 ]
THG in Doubly-Resonant Cavities


Not easy to make at micro-scale
— must precisely tune $\omega_3 / \omega_1$
— materials must be ok at $\omega_1$ and $3\omega_1$

But … what if we could do it?
… what are the consequences?

e.g. ring resonator with proper geometry
Coupled-mode Theory for THG
third harmonic generation

\[
\frac{da_1}{dt} = \left( i\omega_1 \left( 1 - \alpha_{11} |a_1|^2 - \alpha_{13} |a_3|^2 \right) - \frac{1}{\tau_1} \right) a_1 - i\omega_1 \beta_1 (a_1^*)^2 a_3 + \sqrt{\frac{2}{\tau_{s,1}}} s_+
\]

\[
\frac{da_3}{dt} = \left( i\omega_3 \left( 1 - \alpha_{33} |a_3|^2 - \alpha_{31} |a_1|^2 \right) - \frac{1}{\tau_3} \right) a_3 - i\omega_3 \beta_3 a_1^3 + \sqrt{\frac{2}{\tau_{s,3}}} s_+
\]

[ Rodriguez et al. (2007) ]
\( \alpha = 0: \) Critical Power for Efficient THG

third-harmonic generation in doubly-resonant \( \chi^{(3)} \) (Kerr) cavity

\[ P_{\text{crit}} \sim \frac{V}{Q^2} \]

\( \sim \) mW for Si, telecom bandwidth & \( \lambda \)-scale cavity

reflection at \( \omega_1 \)

[ Rodriguez et al. (2007) ]
Detuning for Kerr THG

[ Hashemi et al (2008) ]

because of SPM/XPM, the input power changes resonant $\omega$

... compensate by pre-shifting resonance so that at $P_{\text{in}} = P_{\text{crit}}$
we have $\omega_3 = 3 \omega_1$
Stability and Dynamics?

brief review

Steady state-solution: $a_1$ oscillating at $\omega_1$, $a_3$ at $\omega_3$

— rewrite equations in terms of $A_1 = a_1 \ e^{i\omega_1 t}$
  $A_3 = a_3 \ e^{i\omega_3 t}$

then steady state = $A_1$, $A_3$ constant = fixed-point

cartoon phase space ($A_1$, $A_3$ are actually complex)
for simplicity, assume SPM = XPM coefficients:

\[ \alpha_{11} = \alpha_{33} = \alpha_{13} = \alpha_{31} = \alpha \]
THG Stability Phase Diagram

[ Hashemi et al (2008) ]

\[
\frac{Q_3}{3Q_1} = \frac{\tau_3}{\tau_1}
\]

- unstable 100%-efficiency
- lower-efficiency stable solutions

SPM+XPM / THG
Bifurcation with Input Power

$P_{\text{in}} / P_{\text{crit}}$

THG efficiency

[Hashemi et al (2008) ]
Limit Cycles

Steady state-solution: \( a_1 \) oscillating at \( \omega_1 \), \( a_3 \) at \( \omega_3 \)
— rewrite equations in terms of

\[
A_1 = a_1 e^{i\omega_1 t}
\]

\[
A_3 = a_3 e^{i\omega_3 t}
\]

then steady state = \( A_1, A_3 \) constant = fixed-point

---

cartoon phase space (\( A_1, A_3 \) are actually complex)
Stability Phase Diagram

[ Hashemi et al (2008) ]

\[ \frac{Q_3}{3Q_1} = \frac{\tau_3}{\tau_1} \]

- unstable 100\%-efficiency —
- lower-efficiency stable solutions
  + limit cycles

**Diagram labels:**
- singly-stable
- triply-stable
- doubly-stable
An Optical Kerr-THG Oscillator

[ analogous to self-pulsing in SHG; Drummond (1980) ]

[ Hashemi et al (2008) ]

\[
T \sim 3 \times 10^4 (2\pi/\omega)
\]