18.325 Problem Set 4
Due Thursday, 27 October 2005.

Problem 1: Group velocity and material dispersion

Suppose that we have a dispersive medium with some frequency-dependent dielectric constant \( \varepsilon(\omega) \). (We still assume \( \varepsilon(\omega) > 0 \) and real, however.) In this case, our eigen-equation becomes a “self-consistent” equation \( \Theta |H\rangle = \frac{\omega^2}{\varepsilon} |H\rangle \), since the eigen-operator \( \Theta \) depends on the eigenvalue \( \omega \).

(a) Explain why \( \omega \) is still real, and why first-order perturbation theory (and thus the Hellman-Feynman theorem) still holds for such a self-consistent Hermitian eigen-like problem. Give a property of Hermitian eigen problems that is not true any longer for the self-consistent problem.

(b) Apply the Hellman-Feynman theorem to the self-consistent problem \( \Theta_k |H_k\rangle = \frac{\omega^2}{\varepsilon} |H_k\rangle \) in order to get an expression for the group velocity \( v_g = \frac{d\omega}{dk} \). Note that you will have to use the chain rule, because varying \( k \) causes \( \Theta_k \) to change directly and it also causes \( \varepsilon \) to change due to the change in \( \omega \): \( \frac{d\Theta_k}{dk} + \frac{\partial \Theta_k}{\partial \varepsilon} \frac{d\varepsilon}{dk} \cdot v_g \).

(c) Show that your \( v_g \) is the ratio of mean flux to mean energy density, where the energy density is now the “dispersive” energy density \( \frac{1}{8\pi} \left( \frac{d\omega}{dx} \right)^2 |E|^2 + |H|^2 \), as derived e.g. in Jackson (Classical Electrodynamics).

Problem 2: Dispersion

Derive the width of a narrow-bandwidth Gaussian pulse propagating in 1d \((x)\) in a dispersive medium, as a function of time, in terms of the dispersion parameter \( D = \frac{2\pi c}{\varepsilon_0 \lambda^2} \frac{dv}{d\omega} = -\frac{2\pi c}{\lambda^2} \frac{d^2 k}{d\omega} \) as defined in class. That is, assume that we have a pulse whose fields can be written in terms of a Fourier transform of a Gaussian distribution:

\[
\text{fields} \sim \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(k-k_0)^2/2\sigma^2} e^{ikx} e^{i(\omega_0 - \omega)t},
\]

with some width \( \sigma \) and central wavevector \( k_0 \gg \sigma \). Expand \( \omega \) to second-order in \( k \) around \( k_0 \) and compute the inverse Fourier transform to get the spatial distribution of the fields, and define the “width” of the pulse in space as the standard deviation of the fields.

(a) Compute and plot the TM projected band diagram for the quarter-wave stack with \( \varepsilon \) of 12 and 2.25. That is, plot \( \omega(k_x) \) for several bands, first with \( k_x = 0 \), then \( k_x = 0.1 \), then 0.2, then ... then 0.5, and interpolate intermediate \( k_x \) to shade in the “continuum” regions of the projected bands. Verify that the extrema of these continua lie at either \( k_x = 0 \) or \( k_x = 0.5 \) (in units of \( 2\pi/a \)), i.e. at the B.Z. edges.

(b) Compute and plot the TE projected band diagram as well.

(c) Find the size \( \Delta \omega/\omega_{\text{mid}} \) of the range of omi-directional reflection (from an interface with air). Vary the layer thicknesses \( d_1 \) or \( d_2 \) slightly and see how the size of the omni-directional gap changes..do the quarter-wave thicknesses lead to the maximum omni-directional gap?  

1. Strictly speaking, any dispersive \( \varepsilon \) must be complex to satisfy causality (see e.g. Jackson, Classical Electrodynamics or Google “Kramers-Kronig relations”), but for weak dispersion we can neglect the absorption loss (imaginary part) to a first approximation.

2. You might want to add a “\( kx \)” parameter via “(define-param kx 0)” so that you can change \( k_x \) from the command line with “mpb kx=0.3 ...”.

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Problem 4: Fabry-Perot Waveguides

Modify the MPB defect1d.ctl file from problem set 3 to compute the defect mode as a function of \( k_y \) (for \( k_x = 0 \)).

(a) Changing a single \( \varepsilon_2 \) layer by \( \Delta \varepsilon = 4 \), with an \( N = 20 \) supercell, plot the first 80 bands as a function of \( k_y \) for some reasonable range of \( k_y \). Overlay your TM projected band diagram from problem 3, above, to show that the bands fall into two categories: modes that fall within the projected “continuum” regions from problem 3, and discrete guided bands that lie within the empty spaces. (If there are any bands just outside the edge of the continuum region, increase the supercell size to check whether those bands are an artifact of the finite size.) Plot the fields for the guided bands (a couple of nonzero \( k_y \) points will do) to show that they are indeed localized.

(b) Modify the structure and plot the new band diagram(s) (if necessary) to give examples of:

   (i) “index-guided” bands that do not lie within a band gap

   (ii) a band in the gap that intersects a continuum region

   (iii) a band in the gap that asymptotically approaches, but does not intersect, the continuum regions