18.303 Problem Set 1

Due Friday, 12 September 2014.

Note: For computational (Julia-based) homework problems in 18.303, turn in with your solutions a printout of any commands used and their results (please edit out extraneous/irrelevant stuff), and a printout of any graphs requested; alternatively, you can email your notebook (.ipynb) file to the grader cjfan@math.mit.edu. Always label the axes of your graphs (with the xlabel and ylabel commands), add a title with the title command, and add a legend (if there are multiple curves) with the legend command. (Labelling graphs is a good habit to acquire.) Because IJulia notebooks let you combine code, plots, headings, and formatted text, it should be straightforward to turn in well-documented solutions.

Problem 1: 18.06 warmup

Here are a few questions that you should be able to answer based only on 18.06:

(a) Suppose that \( B \) is a Hermitian positive-definite matrix. Show that there is a unique matrix \( \sqrt{B} \) which is Hermitian positive-definite and has the property \( (\sqrt{B})^2 = B \). (Hint: use the diagonalization of \( B \).)

(b) Suppose that \( A \) and \( B \) are Hermitian matrices and that \( B \) is positive-definite.
   (i) Show that \( B^{-1}A \) is similar (in the 18.06 sense) to a Hermitian matrix. (Hint: use your answer from above.)
   (ii) What does this tell you about the eigenvalues \( \lambda \) of \( B^{-1}A \), i.e. the solutions of \( B^{-1}Ax = \lambda x \)?
   (iii) Are the eigenvectors \( x \) orthogonal?
   (iv) In Julia, make a random \( 5 \times 5 \) real-symmetric matrix via \( A = \text{rand}(5,5); A = A + A' \) and a random \( 5 \times 5 \) positive-definite matrix via \( B = \text{rand}(5,5); B = B' * B \) ... then check that the eigenvalues of \( B^{-1}A \) match your expectations from above via \( \text{eigvals}(B' \backslash A) \) (this will give an array \( \text{lambda} \) of the eigenvalues and a matrix \( X \) whose columns are the eigenvectors).
   (v) Using your Julia result, what happens if you compute \( C = X^T BX \) via \( C = X' * B * X \)? You should notice that the matrix \( C \) is very special in some way. Show that the elements \( C_{ij} \) of \( C \) are a kind of “dot product” of the eigenvectors \( i \) and \( j \), but with a factor of \( B \) in the middle of the dot product.

(c) The solutions \( y(t) \) of the ODE \( y'' - 2y' - cy = 0 \) are of the form \( y(t) = C_1 e^{(1 + \sqrt{1+c})t} + C_2 e^{(1 - \sqrt{1+c})t} \) for some constants \( C_1 \) and \( C_2 \) determined by the initial conditions. Suppose that \( A \) is a real-symmetric \( 4 \times 4 \) matrix with eigenvalues \( 3, 8, 15, 24 \) and corresponding eigenvectors \( \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_4 \), respectively.
   (i) If \( \mathbf{x}(t) \) solves the system of ODEs \( \frac{d^2 \mathbf{x}}{dt^2} - \frac{2}{\sqrt{1+c}} \mathbf{x} = A \mathbf{x} \) with initial conditions \( \mathbf{x}(0) = \mathbf{a}_0 \) and \( \mathbf{x}'(0) = \mathbf{b}_0 \), write down the solution \( \mathbf{x}(t) \) as a closed-form expression (no matrix inverses or exponentials) in terms of the eigenvectors \( \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_4 \) and \( \mathbf{a}_0 \) and \( \mathbf{b}_0 \). (Hint: expand \( \mathbf{x}(t) \) in the basis of the eigenvectors with unknown coefficients \( c_1(t), \ldots, c_4(t) \), then plug into the ODE and solve for each coefficient using the fact that the eigenvectors are)
   (ii) After a long time \( t \gg 0 \), what do you expect the approximate form of the solution to be?
Problem 2: Les Poisson, les Poisson

In class, we considered the 1d Poisson equation $\frac{d^2}{dx^2} u(x) = f(x)$ for the vector space of functions $u(x)$ on $x \in [0, L]$ with the “Dirichlet” boundary conditions $u(0) = u(L) = 0$, and solved it in terms of the eigenfunctions of $\frac{d^2}{dx^2}$ (giving a Fourier sine series). Here, we will consider a couple of small variations on this:

(a) Suppose that we we change the boundary conditions to the periodic boundary condition $u(0) = u(L)$.

(i) What are the eigenfunctions of $\frac{d^2}{dx^2}$ now?

(ii) Will Poisson’s equation have unique solutions? Why or why not?

(iii) Under what conditions (if any) on $f(x)$ would a solution exist? (You can restrict yourself to $f$ with a convergent Fourier series.)

(b) If we instead consider $\frac{d^2}{dx^2} v(x) = g(x)$ for functions $v(x)$ with the boundary conditions $v(0) = v(L) + 1$, do these functions form a vector space? Why or why not?

(c) Explain how we can transform the $v(x)$ problem of the previous part back into the original $\frac{d^2}{dx^2} u(x) = f(x)$ problem with $u(0) = u(L)$, by writing $u(x) = v(x) + q(x)$ and $f(x) = g(x) + r(x)$ for some functions $q$ and $r$. (Transforming a new problem into an old, solved one is always a useful thing to do!)

Problem 3: Finite-difference approximations

For this question, you may find it helpful to refer to the notes and reading from lecture 3. Consider a finite-difference approximation of the form:

$$u'(x) \approx \frac{-u(x + 2\Delta x) + c \cdot u(x + \Delta x) - c \cdot u(x - \Delta x) + u(x - 2\Delta x)}{d \cdot \Delta x}.$$ 

(a) Substituting the Taylor series for $u(x + \Delta x)$ etcetera (assuming $u$ is a smooth function with a convergent Taylor series, blah blah), show that by an appropriate choice of the constants $c$ and $d$ you can make this approximation fourth-order accurate: that is, the errors are proportional to $(\Delta x)^4$ for small $\Delta x$.

(b) Check your answer to the previous part by numerically computing $u'(1)$ for $u(x) = \sin(x)$, as a function of $\Delta x$, exactly as in the handout from class (refer to the notebook posted in lecture 3 for the relevant Julia commands, and adapt them as needed). Verify from your log-log plot of the $|\text{errors}|$ versus $\Delta x$ that you obtained the expected fourth-order accuracy.