Finding a Maximum Cardinality Matching in Bipartite Graphs

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Instructions

Each of you needs a kidney but doesn’t have a willing donor who is a match (same blood type).

Take 5 minutes to find a partner with whom you can swap donor kidneys.
Our Goal:

• Make an algorithm to find the largest cardinality matching (most sets of partners) in a bipartite graph.

• *Teaser: This algorithm will also find the minimum size vertex cover of the graph!*
Helpful Definitions (1/2)

- **Bipartite Graph**: a graph whose vertices can be split into two disjoint sets (- and +) such that every edge connects a vertex in - to one in +

- **Matching**: a set of edges without common vertices

  - **Maximum Cardinality Matching**: largest # of edges
Helpful Definitions (2/2)

• An **alternating path** with respect to $M$ alternates between edges in $M$ and in $E-M$.

• An **augmenting path** with respect to $M$ is an alternating path with first and last vertices exposed.
Helpful Definitions (2/2)

• An **alternating path** with respect to $M$ alternates between edges in $M$ and in $E-M$

• An **augmenting path** with respect to $M$ is an alternating path with **first and last vertices exposed**
True or False?

1. This is a matching:
True or False?

1. This is a matching:

FALSE
True or False?

2. This is a matching:
2. This is a matching: 

TRUE
True or False?

3. This is a maximum cardinality matching:
True or False?

3. This is a maximum cardinality matching:

FALSE
True or False?

4. This is an alternating path:
True or False?

4. This is an alternating path:

TRUE
True or False?

5. This is an **augmenting path**:
True or False?

5. This is an **augmenting path**: TRUE
True or False?

6. This is an **augmenting path:**

![Diagram of a network with a green dashed line as an augmenting path]
True or False?

6. This is an **augmenting path:**

TRUE
Augmenting a Matching

\[ M = \{(1,6),(2,7)\} \]
Augmenting a Matching

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\[ M = \{(1,6), (2,7)\} \]

\[ P = \{(5,7), (7,2), (2,9)\} \]
Augmenting a Matching

$M = \{(1,6),(2,7)\}$  

$M' = \{(1,6),(2,9),(5,7)\}$

$P = \{(5,7), (7,2),(2,9)\}$
Algorithm

1 – Start with any matching $M$ (let’s say $M = {}$)
2 – As long as there exists an **augmenting path** with respect to $M$:
   3 – Find augmenting path $P$ with respect to $M$
   4 – Augment $M$ along $P$: $M' = M \Delta P$
5 – Replace $M$ with the new $M'$
Proof: A matching is maximum iff there are no augmenting paths
(by contradiction)

» If we have some augmenting path P wrt M:
   \[ M' = M \Delta P, \text{ and } |M'| > |M| \]
   (so \( M \) couldn’t have been maximum)

« If \( M \) isn’t maximum, there must be some augmenting path Q such that
   \[ Q = M \Delta M^* \] (where \( M^* \) is a maximum matching)
If $M$ isn’t maximum, there must be some augmenting path $Q$ such that $Q = M \Delta M^*$

- $Q$ has more/fewer/equal edges from $M^*$ than from $M$
If $M$ isn’t maximum, there must be some augmenting path $Q$ such that $Q = M \Delta M^*$

• $Q$ has **more/fewer/equal** edges from $M^*$ than from $M$
If $M$ isn’t maximum, there must be some augmenting path $Q$ such that $Q = M \Delta M^*$

• $Q$ has more/fewer/equal edges from $M^*$ than from $M$

• Each vertex is incident to ___________ edge(s) in $M^Q$ and ___________ edge(s) in $M^*^Q$.
If $M$ isn’t maximum, there must be some augmenting path $Q$ such that $Q = M \Delta M^*$

- $Q$ has more/fewer/equal edges from $M^*$ than from $M$
- Each vertex is incident to at most one edge(s) in $M^Q$ and at most one edge(s) in $M^*^Q$.

If $M$ isn’t maximum, there must be some augmenting path $Q$ such that $Q = M \Delta M^*$

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If $M$ isn’t maximum, there must be some augmenting path $Q$ such that $Q = M \Delta M^*$

- $Q$ has more/fewer/equal edges from $M^*$ than from $M$

- Each vertex is incident to \textbf{at most one} edge(s) in $M \wedge Q$ and \textbf{at most one} edge(s) in $M^* \wedge Q$

- Therefore, $Q$ is composed of cycles and paths that alternate between edges from $M$ and $M^*$
If $M$ isn’t maximum, there must be some augmenting path $Q$ such that $Q = M \triangle M^*$

- $Q$ has more/fewer/equal edges from $M^*$ than from $M$
- Each vertex is incident to **at most one** edge(s) in $M \wedge Q$ and **at most one** edge(s) in $M^* \wedge Q$
- Therefore, $Q$ is composed of cycles and paths that alternate between edges from $M$ and $M^*$
- There must be some path with more edges from $M^*$ than from $M$. This is an **augmenting path**.
Augmenting a Matching

\[ M = \{(1,6),(2,7)\} \]

1 – Direct all edges in the matching from B to A

[Diagram showing the directed edges and vertices]
Augmenting a Matching

\[ M = \{(1,6),(2,7)\} \]

1. Direct all edges in the matching from B to A, and all edges not in the matching from A to B.

2. Create a node \( s \) that connects to all exposed vertices in set A.

3. Do a Breadth First Search to find an exposed vertex in set B from node \( s \).
Augmenting a Matching

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Augmenting a Matching

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Try one with a partner!
Algorithm

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König’s Theorem (1931)

For any bipartite graph, the maximum size of a matching is equal to the minimum size of a vertex cover
Kidney Transplants
Kidney Transplants
Questions?