Chapter 2: Problem 10

This is the ‘discrete metric’ on a set. Certainly \( d : X \times X \rightarrow [0, \infty) \) is well defined and \( d(x, y) = 0 \) iff \( x = y \). Symmetry, \( d(x, y) = d(y, x) \), is immediate from the definition and the triangle inequality

\[
(1) \quad d(x, y) \leq d(x, z) + d(z, y) \quad \forall \, x, y, z \in X
\]

follows from the fact that the right hand side is always equal to 0, 1 or 2 and the LHS is 0 or 1 and if the LHS vanishes then \( x = y = z \) and the RHS also vanishes.

All subsets are open, since if \( E \subset X \) and \( p \in E \) then \( x \in X \) and \( d(x, p) < 1 \) implies \( x = p \) and hence \( x \in E \). Since the complements of open sets are closed it follows that all subsets are closed. The only compact subsets are finite. Indeed if \( E \subset X \) is compact then the open balls of radius 1 with centers in \( E \) cover \( E \) and each contains only one point of \( E \) so the existence of a finite subcover implies that \( E \) itself is finite.

Chapter 2: Problem 12

We are to show that \( K = \{ 1/n; n \in \mathbb{N} \} \cup \{ 0 \} \) is compact as a subset of \( \mathbb{R} \) directly from the definition of compactness. So, let \( U_a, a \in A, \) be an open cover of \( K \). It follows that \( 0 \in U_{a_0} \) for some \( a_0 \in A \). But since \( U_{a_0} \) is open it contains some ball of radius 1/\( n \) around 0. Thus all the points \( 1/m \in U_{a_0} \) for \( m > n \). For each \( m \leq n \) we can find some \( a_m \in A \) such that \( 1/m \in U_{a_m} \), since the \( U_a \) cover \( K \). Thus we have found a finite subcover

\[
(2) \quad K \subset U_{a_0} \cup U_{a_1} \cup \cdots \cup U_{a_n}
\]

and if follows that \( K \) is compact.

Chapter 2: Problem 16

Here \( \mathbb{Q} \) is the metric space, with \( d(p, q) = |p - q| \), the ‘usual’ metric. Set

\[
(3) \quad E = \{ p \in \mathbb{Q}; 2 < p^2 < 3 \}.
\]

Suppose \( x \) is a limit point of \( E \) as a subset of the rationals. Then we know that \( (x - \epsilon, x + \epsilon) \cap E \) is infinite for each \( \epsilon > 0 \). Regarding \( x \) as a real number it follows that \( x \in [2^{1/2}, 3^{1/2}] \). Since we know the end points are not rational and by assumption \( x \in \mathbb{Q} \) it follows that \( x \in E \). Thus \( E \) is closed. Certainly \( E \) is bounded since \( p \in E \) implies \( |p| < 3 \).

To see that \( E \) is not compact, recall that if it were compact as a subset of \( \mathbb{Q} \) it would be compact as a subset of \( \mathbb{R} \) by Theorem 2.33. Since it is not closed as a subset of \( \mathbb{R} \) it cannot be compact. Alternatively, for a direct proof of non-compactness, take the open cover given by the open sets \( \{ x \in \mathbb{Q}; |p - 2^{1/2}| > 1/n \} \). This can have no finite subcover since \( E \) contains points arbitrarily close to the real point \( \sqrt{2} \).

Yes \( E \) is open in \( \mathbb{Q} \) since it is of the form \( (\sqrt{2}, \sqrt{3}) \cap \mathbb{Q} \) where \( G = (\sqrt{2}, \sqrt{3}) \subset \mathbb{R} \) is open, so Theorem 2.30 applies.
Chapter 2: Problem 22

We need to show that the set of rational points, \( \mathbb{Q}^k \), is dense in \( \mathbb{R}^k \). We can use the fact that \( \mathbb{Q} \subset \mathbb{R} \) is dense. Thus, given \( \epsilon > 0 \) and \( x = (x_1, \ldots, x_k) \in \mathbb{R}^k \) there exists \( p \in \mathbb{Q} \) such that \( |x_l - p_l| \leq \epsilon/k \) for each \( l = 1, \ldots, k \). Thus, as points in \( \mathbb{R}^k \),
\[
|x - p| \leq \sum_{l=1}^{k} |x_l - p_l| < \epsilon.
\]
This shows that \( \mathbb{R}^k \) is separable since we know that \( \mathbb{Q}^k \) is countable.

Chapter 2: Problem 23

We are to show that a given separable metric space, \( X \), has a countable base. The hint is to choose a countable dense subset \( E \in X \) and then to consider the collection, \( B \), of all open subsets of \( X \) of the form \( B(x, 1/n) \) where \( x \in E \) and \( n \in \mathbb{N} \). This is a countable union, over \( \mathbb{N} \), of countable sets so is countable. Now, we need to show that this is a base. So, suppose \( U \subset X \) is a given open set. If \( x \in U \) then for some \( m = m_x > 0 \), \( B(x, 1/m) \subset U \), since it is open. Also, by the density of \( E \) in \( X \) there exists some \( e_x \in E \) with \( |x - e_x| < 1/2m \) But then \( y \in B(e_x, 1/2m) \) implies \( d(x, y) < d(x, e_x) + d(e_x, y) < 1/2m + 1/2m = 1/m \). Thus \( B(e_x, 1/2m) \subset U \). It follows that
\[
U = \bigcup_{x \in U} B(e_x, 1/2m_x).
\]
Thus \( U \) is written as a union of the elements of \( B \) which is therefore an open base.

Chapter 2: Problem 25

We wish to show that a given compact metric space \( K \) has a countable base. As the hint says, for each \( n \in \mathbb{N} \) consider the balls of radius \( 1/n \) around each of the points of \( K \):
\[
K \subset \bigcup_{x \in K} B(x, 1/n)
\]
since each \( x \in K \) is in one of these balls at least. Now the compactness of \( K \) implies that there is a finite subcover, that is there is a finite subset \( C_n \subset K \), for each \( n \), such that
\[
K \subset \bigcup_{p \in C_n} B(p, 1/n).
\]
Now, set \( E = \bigcup_{n \in \mathbb{N}} C_n \). This is countable, being a countable union of finite sets. Now it follows that \( C \) is dense in \( K \). Indeed given \( x \in K \) and \( \epsilon > 0 \) there exists \( n \in \mathbb{N} \) with \( 1/n < \epsilon \) and from (7) a point in \( p \in C_n \subset C \) with \( |x - p| < 1/n < \epsilon \). Thus \( K = \overline{C} \) and it follows that \( K \) is separable; from \#23 it follows that \( K \) has a countable open base.

Alternatively one can see directly that the \( B(p, 1/n), p \in C_n, \) form an open base.

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