Start to put some of these computations together!
We decompose forms near the boundary in a conic manifold as

\( U = x^k u_t + x^{k-1} dx \wedge u_n \) \quad k\text{-form}

where \( u_t, u_n \) are tangent (but \( x \)-dependent) forms of degree \( k, k-1 \). In terms of these decompositions

\[
\mathbf{d} = \begin{pmatrix}
-\frac{1}{x} \frac{1}{dt} & \frac{1}{dx} + \frac{k}{x} \\
0 & -\frac{1}{x} \frac{1}{dt}
\end{pmatrix}
\begin{pmatrix}
u_t \\
u_n
\end{pmatrix}
\]

The \( x \)-factor in (1) is designed so that for the model conic metric

\( g_0 = dx^2 + x^2 h_0 (y, dy) \)

\[
\langle U, U \rangle_k = \langle u_t, u_t \rangle_{k-1} + \langle u_n, u_n \rangle_{k-1}
\]

in terms of the tangent bundle metric near \( x = 0 \).

The definition of \( \mathcal{F} \)

\[
\int \langle \mathcal{F}, U \rangle_k \, dg_0 = \int \langle \psi, \mathcal{F} \rangle_{k-1} \, dg_0
\]
\[ \int \int \left\{ \frac{\partial}{\partial t} \varphi_n + \left( \frac{1}{x} + \frac{k-1}{x} \right) \varphi_t, \ u_n \right\} dx \, dt \]

\[ = \int \int \left\{ \left( \frac{1}{x} \right) \varphi_t, \ u_t \right\} dx \, dt \]

\[ = \int \int \left\{ \left( \frac{\partial}{\partial n} \left( \varphi_n, u_n \right) \right)_{t, k-2} + \left( \frac{\partial}{\partial t} \varphi_t, u_t \right)_{t, k-1} \right\} dx \, dt \]

Thus,

\[ f = \begin{pmatrix} -\frac{1}{2} \delta_t & 0 \\ -\frac{1}{x} \frac{\partial}{\partial n} - \frac{n-k}{x} & \frac{1}{x} \delta_t \end{pmatrix} \]

Combining, we have

\[ d + f = \begin{pmatrix} -\frac{1}{x} (d + \delta) \varphi_t & \frac{1}{x} \delta_t + \frac{k}{x} \end{pmatrix} \]

Now, show for the index multi \((d + f) u = 0\) with \((d + f) (\varphi u) = u + C (x, \lambda)\)
$u_{n,m} = \int x^{s_1} \phi(x) \, dx$

Solve for

$$
\begin{pmatrix}
-(d_t + d_k) & -is + k \\
is - (n-k) & (d_t + d_k)
\end{pmatrix}
\begin{pmatrix}
u_{n,m} \\
u_{t,m}
\end{pmatrix}
= \text{eigenvalue}\ \frac{c}{\Xi(s_i + 1)}
$$

We can discuss the full unraveling of the operator but for the moment let me point out that the kernel does not act as a vector except for a discrete set of poles for the multiplicity of each.

So $\begin{pmatrix} u_{n,m} \\ u_{t,m} \end{pmatrix}$ is nonzero for $n$ as desired.

What are the poles, especially for the step

$$\frac{n}{2} < \Im s < \frac{n}{2} + 1 ?$$

As for $d^{1/2}$ if $du = 0, du = \infty$?
there is a slightly more explicit form of what we can ask.

Namely, for a semi-variational compact manifold with boundary, $d + s : D_B \to L^2$ is Fredholm and self-adjoint for $B = A, R$ if

$$H^k_{A-L^2}(X) \cong H^k_{A-H^0}(X) \cong \{ u + D_A; (d + s)u = 0 \}.$$

Also

1. If $n$ is even, $D_A = D_R$
2. If $n$ is odd, $D_A = D_R \iff H^k(2\pi) = \{ 0 \}$.

Exercise (straightforward) show that

$$*D_A = D_R$$

always.

Thus, if $n$ is even, we have Poincaré duality,

$$*H^k_{A-L^2} = H^{n-k}_{A-L^2}.$$
if $n$ is odd then $\nabla \theta$ not held.

Proof of (1) given the rest of the theorem.
As clearly remarks we do this using the Hodge decomposition:

$$L^2_c(X, \Lambda^k) = H^k_B \oplus d D^{k-1}_B \oplus \delta D^k_B \oplus \text{H}.$$ \hfill (8)

Let's prove the first, for $B = A$ for the start.

$$u, v \in D_A \Rightarrow (du, dv)_{L^2_j} = \lim_{j \to \infty} (du, dv_j) = \lim_{j \to \infty} (d^2 u, v_j) = 0$$

Since $u \in D_A$ then $\exists v_j \in C^\infty(X, \Lambda^k), v_j \to u$ in $L^2$, $dv_j \to dv$ in $L^2$. Then the second two terms on the right in (8) are all gone.

$$(d + \delta)D_A = d D_A \oplus \delta D_A$$

is closed hence built an exact. By second selfadjoint $(d + \delta)D_A$ is a selfadjoint.
The identification (1) now follows directly from (8). Namely we define
\[(P) \quad \{u \in L^2 : du = 0\} \to H_{A-H_0}^* (\mathbb{R})\]
by projecting into its harmonic part in (8).
Since we know that \(u \in DA \Rightarrow du = \nabla u\), \(u \in H_{A-H_0}^*\)
dro right \(du = 0 \Rightarrow 0\); it follows that \((P)\)
is surjective. If \(u \in L^2(\mathbb{R}^d, \mathbb{R}^d)\) and \(du = 0\) then the Hodge decomposition
\[
\n u = u_H + dV + \delta w
\]
necessarily has \(\delta w = 0\). Indeed, \(dV = 0\) while \(d\delta w = 0\) and

\[
0 = \langle w, d\delta w \rangle = \lim_{|\alpha| \to \infty} \langle w_\alpha, d\delta w \rangle
\]

\[
= \lim_{|\alpha| \to \infty} \langle \delta w_\alpha, w \rangle = \|\delta w\|
\]

where \(w_\alpha \in C^\infty(\mathbb{R}^d, \mathbb{R}^d), w_\alpha \to w\) in \(L^2 \times L^2 \to L^2\). Then,

\[
(C) \quad u = u_H + dV, \quad V \in DA,
\]
so the null space of \((P)\) consists of solutions of
with \( u = d \nu, \forall \in \mathcal{D}_R \). Consider, \( \langle \cdot, \cdot \rangle \) to form:

\[
\langle m_H, d\nu \rangle = \frac{1}{i} \langle w_j, d\nu \rangle = \frac{1}{i} \langle dw_j, w \rangle = \frac{1}{i} \langle dw_j, w \rangle = 0
\]

since as \( w_j \to w \in L^2 \) with \( dw_j \to dw \). Thus, \( u = d\nu \) so the null space of \( (P) \) is

\[
\{ u \in L^2(X; X^k); u = d\nu, \nu \in L^2(X; \Lambda^{k-1}) \}
\]

and we see that \((7)\) is correct.

Exercise 2.4: through the analogous argument and show that:

\[
\{ u \in L^2(X; \Lambda^k); \int_X u = 0 \}
\]

is

\[
H^k \quad \mathcal{R} \quad H^k
\]

\[
\mathcal{N} = H^k \quad (X) = \left\{ u \in \mathcal{D}_R \mid (d + \mathcal{S})u = 0 \right\}.
\]

So, it remains to establish the first half of Theorem F, showing that \( d \mathcal{S} \) is Fredholm as self-adjoint on \( \mathcal{D}_R \) + \( \mathcal{D}_R \).
Now, we return to the question of the regularity of \( u_{1+n} \) satisfying \( Lu = 0, Lu_1 = 0 \) for its initial matrix. First, consider the projection of \( u_{1+n} \) onto the harmonic forms. It follows directly the

\[
(-i\sigma + k) \tilde{v}_{1+n}^H \quad \text{and} \quad (i\sigma - (n-k)) \tilde{v}_{1+n}^H \quad \text{an}
\]

\[
\text{error (with values in } C^\infty \text{ at } \rho \text{ fit decay at } R_0(-\infty)).
\]

So we conclude \( \tilde{v}_{1+n}^H \) can have at most a simple pole

\[
at -i\sigma = -k
\]

(A) \( \tilde{v}_{1+n}^H \) can have at most a simple pole

\[
at -i\sigma = -(n-k)
\]

(A) \( \tilde{v}_{1+n}^H \) can have at most a simple pole

\[
at -i\sigma = -k = -\frac{n}{2} + \frac{1}{2} \quad \text{if } k = \frac{n}{2} - \frac{1}{2}
\]

if \( n \) is odd

\[
at -i\sigma = -(n-k) = -\frac{n}{2} + \frac{1}{2} \quad \text{if } k = \frac{n}{2} + \frac{1}{2}
\]

if \( n \) is even

\[
at -i\sigma = -(n+1) = -\frac{n}{2} + 1 \quad \text{if } k = \frac{n}{2} - 1
\]

Note that these values are in the critical strip

\[-\frac{n}{2} < \text{Re} \sigma < \frac{n}{2} + 1 \quad \text{for}
\]

(A) if \( n \) is odd

\[
-\sigma = -k = -\frac{n}{2} + \frac{1}{2} \quad \text{if } k = \frac{n}{2} - \frac{1}{2}
\]

-\( -\sigma = -(n-k) = -\frac{n}{2} + \frac{1}{2} \quad \text{if } k = \frac{n}{2} + \frac{1}{2}
\]

-\( -\sigma = -(n+1) = -\frac{n}{2} + 1 \quad \text{if } k = \frac{n}{2} - 1
\]
Next observe that
\[ \frac{d}{dt} u_t = 0 \Rightarrow u_t = 0 \]

\[ \left\{ \begin{array}{l}
\frac{d}{dt} u_n = 0 \Rightarrow u_n = 0 \\
\end{array} \right. \]

where we note \( u = u^t + d u + \delta u \)

for the Hodge decomposition for \( h_0 \) on the boundary.

This we can write the remaining conditions as

\[ \begin{align*}
-\frac{d}{dt} u_n & + \left( -i \sigma + k \right) u_n \\
& = \text{entic} \\
\end{align*} \]

Apply \( \frac{d}{dt} \) to the 1st and note the second

we find \( k_{(k)} \)

\[ -\frac{d}{dt} u_n + \left( -i \sigma + k \right) u_n = 0 \quad \text{over} \]

\[ -\frac{d}{dt} u_n - \left( -i \sigma + k \right) \left( i \sigma - (n-k) \right) u_n \quad \text{is} \]

A derive \( u_n = \frac{d}{dt} = A \) so

\[ A - \left( -i \sigma + k \right) \left( -i \sigma + n-k \right) \]

is constant.
The uneven, \((\Delta - (-i \sigma + \frac{n}{2})^2 + (\frac{n}{2} - k)^2)^{-1}\)

is here for non-uniform, but only (strict) fits of the points when

\[ (-i \sigma + \frac{n}{2})^2 = (\frac{n}{2} - k)^2 + \delta_j^2 \]

where \(\delta_j\) is one of the constant eigenvalues of \(\Delta\) (so it is strictly positive):

\[
-i \sigma A_\phi = -\frac{n}{2} \pm \sqrt{\left(\frac{n}{2} - k\right)^2 + \delta_j^2}.
\]

So, line on both on the finer regularizing axis strictly done. Where

\[-\frac{n}{2} + \left|\frac{n}{2} - k\right| \quad \text{or} \quad \text{strictly below} \quad -\frac{n}{2} - \left|\frac{n}{2} - k\right|\]

The latter as it is the region where we know \(\mathcal{U}_{n,m}\) to be homomorphic so the only possible places are at

\[(C_0) \quad -i \sigma = -\frac{n}{2} + \left|\frac{n}{2} - k\right| + e_j \]

\[e_j = \sqrt{\left(\frac{n}{2} - k\right)^2 + \delta_j^2} - \left|\frac{n}{2} - k\right| \geq 0\]

Thus an only as the critical step if

\[n \in \mathbb{N}, \quad k = \frac{n}{2} \pm \frac{1}{2}, \quad 0 < e_j < \frac{1}{2}\]