Divac and other differential operators

Let me quickly review the basic definitions of forms and vector fields, starting from a $C^\infty$ manifold. As before, the idea is doing this so to help with subsequent generalizations.

A $C^\infty$ manifold, $X$, for the moment without curves if you wish, comes equipped with a space of smooth functions, $C^\infty(X)$. A point $p\in X$ can be recoved from its defining ideal

$$J_p = \{u\in C^\infty(X) : u(p) = 0\}.$$ 

If we let $J_p^2 \subseteq C^\infty(X)$ be the finite span of products of elements of $J_p$

$$J_p^2 = \{u\in C^\infty(X) : u = \sum_{i=1}^n f_i g_i, \quad f_i, g_i \in J_p\}$$

then

$$T_p^* X = J_p / J_p^2$$
as the tangent space at \( p \). If \( u \in C^0(X) \) and \( u - m \in I_p \) so there is a well-defined element (1)

\[
d u(p) \in T^*_p X.
\]

Exercise: Show that if \( S_1, \ldots, S_X \) are local coordinates at \( p \) then \( d S_1, \ldots, d S_X \) as a basis for \( T^*_p X \); conclude that

\[
T^*_X = \bigcup_{p \in X} T^*_p X
\]

as a (real) vector bundle over \( X \).

Note that the local coordinates in \( T^*_U = \bigcup_{p \in U} T^*_p X \) induced by local coordinates \( S_1, -i S_2, \ldots, S_X \) on \( U \subset X \) are given by \( (S_1, -i S_2, \ldots, S_X) \) where

\[
S = \sum_{j=1}^n S_j \cdot d \beta_j, \quad S \in T^*_p X.
\]

The \( C^0 \) structure on \( T^*_X \) is endowed with (1), namely

(2) \[
d : C^0(X) \to C^0(X; T^*_X)
\]

the map on the right can use the notation for smooth
section of \( T^*X \), this is the basic example of a (geometric) differential operator.

The usual tangent bundle \( TX \) can be defined as the dual of \( T^*X \) or dually in terms of derivations. That is

\[
T_pX = \{ v : C^0(X) \to \mathbb{R} ; v \text{ is linear and } v(fg) = f(p) v(g) + v(f) g(p) \}.
\]

Exercise: Show that \( T_pX \cong (T^*_pX)^* \) with the identification being given by a pairing

\[
T_pX \times T^*_pX \ni (v,s) \mapsto v(f), \quad f \in \mathcal{F}_p, \quad \|f\| = \sum_{i=1}^{n} |f_i| \in T^*_pX.
\]

A section \( \nu \in C^0(X, TX) \) will define a linear map

\[
\nu : C^0(X) \to C^0(X), \quad \nu f(p) = \nu_p f.
\]

By definition a linear differential operator, will have a coefficient acting a functions in just a combination of such vector fields.
\[ P : C^0(\mathbb{R}) \to C^0(\mathbb{R}), \]

\[ P_u = \sum_{\text{finite}} V_{k_a} \cdots V_{k_1} u \quad (X \text{ compact}) \]

The first ad can, \( k_a \leq 1 \), is just \( P = V + f \), \( V \in C^0(\mathbb{R}^+, TX), f \in C^0(\mathbb{R}) \).

We are interested in differential operators acting on (complex) vector bundles. Let me give a couple of equivalent definitions.

First, we can "reset" to local coordinates. The basis of \( T_pX \) induced by local coordinates \( \{ e_1, \ldots, e_n \} \) in \( \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n} \), where \( \frac{\partial}{\partial x^i} \cdot \frac{\partial}{\partial x^j} = \delta_{ij} \).

The local coordinates form \( f(3) \) as \( \text{let} \)

\[ P_u = \sum_{|\alpha| \leq m} \delta(\alpha) \frac{\partial^\alpha}{\partial x^\alpha} c^\alpha \]

\( \text{using multiindex notation, } \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n \).
\[ \lambda_l = \lambda_1 + \cdots + \lambda_n, \quad D^\lambda = \left( \frac{1}{i} \partial^\lambda \right) = i^{m_1} \frac{\partial^{\lambda_1}}{\partial x_1^{\lambda_1}} \cdots \frac{\partial^{\lambda_n}}{\partial x_n^{\lambda_n}} \]

Thus a differential operator \( P \) is just \( \lambda \)-wise.

\[ P : C^\infty(X) \to C^\infty(X) \] what takes the form (6)

\[ \lambda_1 \log \partial x_1 \cdots \lambda_n \log \partial x_n \] in any local coordinates (not sufficient for this to be true is a covering of \( X \) by charts).

A vector bundle \( E \) has local trivialization \( U \) on \( X \) so covered by coordinate patches \( U_i \) on each of which \( E \) has a basis of smooth sections \( e_i \) - \( \mathbb{R} \).

If \( F \) is another vector bundle then we can find a covering by coordinate charts on which both \( E \) or \( F \) have fixed bases, as an trivial. Then

\[ P : C^\infty(X; E) \to C^\infty(X; F) \]

a linear map, on a differential operator of order

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(\) at most) in if

\[ P \left( \sum_{k=1}^{K} \Phi_k \, \xi_k \right) = \sum_{k=1}^{K} \left( P \xi_k \Phi_k \right) \xi_k \]
when the $P_i$ are of the form (4). An interesting, although essentially useless, result is

Theorem (Reede) $P_i: C^0(X, E) \to C^0(X, F)$ is linear

iff on a different operator if and only if

$u = 0$ on $U \subset X \Rightarrow P_i u = 0$ on $U$

$\forall U \subset X$ then.

Exercise: Let $\text{Diff}^m(X; E, F)$ denote the space of linear differential operators between sections of bundles $E$ and $F$. Show that composition of operators (of order at most $m$)

defines a product

$\text{Diff}^m(X; E, E_3) \cdot \text{Diff}^m(X; E_1, E_2) \subset \text{Diff}^m(X; E_1, E_3)$.

As I said before, we are mostly interested in first order operators, but we continue a little bit further here.
Suppose \( f \in C^0(X) \) and \( u \in C^0(X,E) \) for \( f \in E \),

(17) \( \eta_d = e^{i df} u \in C^0(X,E) \)

Leibniz' formula shows us how to distribute differentials on such a product, namely

\[
D^\alpha (uv) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^{\beta} u \cdot D^{\alpha - \beta} v,
\]

If we think that this means for \( P \), i.e.,

and considering Leibniz rule (6), we see that

(18) \( P(e^{i df} u) = e^{i df} P_d f u \)

Here \( P_d f \) is again a differential operator of the same order, with coefficients depending on \( df \). In fact,

\[
\delta P_d f = \sum_{s=0}^m d^s P_d (s)_f
\]
in a polynomial of degree at most $m-1$. The powers of $t$ come from the derivatives. It is easy to check that $n-3$ 

\[ P(t) \in \text{Diff}^m(X, E, F). \]

Now, if $n = m$, \( \text{Diff}^0(X, E, F) = C^0(X, \text{Hom}(E, F)) \)

is just the space of bundle maps for $E$ to $F$. If $n > m$, $P$ still depends on $f$ but we can see that $P$ is a polynomial in $\text{Diff}^m$ of degree $m$.

\[ \sigma(P) = (\text{Diff}^m, 0). \]

Definition: $\text{Pert}_{\text{Id}} f = \lim_{d \to \infty} e^{-tE} \text{Pert}_{\text{Id}} f e^{tE}$. It is a well-defined polynomial homography of degree $m$, where the fiber of $T^*X$ at value $v$ is $\text{Hom}(E, F)$.

\[ \sigma(P) \in \mathbb{P}^m(T^*X; \text{Hom}(E, F)). \]
In local coordinates, \((5.141)\)

\[
P(\xi^L, \eta^L) = \sum_{\lambda | \lambda = \lambda} \sum_{\lambda' | \lambda' = \lambda'} \left( \phi_{\lambda', \lambda} (3) \Delta_{\lambda'} \right) \xi^{\lambda'}
\]

\[(1)\]

\[
\sigma(P)(8.5) = \sum_{\lambda | \lambda = \lambda} \sum_{\lambda' | \lambda' = \lambda'} \left( \phi_{\lambda', \lambda} (3) \Delta_{\lambda'} \right) \xi^{\lambda'}
\]

Of course, we could just use this as a definition, but then we would have to check coordinate invariance! The abstract-rigorous definition makes it easy to check this.

\[
\sigma(P \phi) = \sigma_{\mu} (P) \cdot \sigma_{\mu} (\phi)
\]

\[(1)\]

\[
P \in \text{Diff}^{\mu}(X, E_2, E_3), \quad \phi \in \text{Diff}^{\mu}(X', E_1, E_2).
\]

This "\(\sigma\)" kills both differential non-commutativity and just keeps the bare non-commutativity, but

The real importance of these "\(\sigma\)" is
Let me recall what a metric on a (real) vector bundle is. It is simply a positive-definite inner product on each fiber, varying smoothly. Thus, a metric \( \langle \cdot, \cdot \rangle_\rho \) on the fiber \( E_\rho \) of \( E \) has to be such that \( \langle e, e' \rangle \) is smooth \( \forall e, e' \in \mathcal{C}(X; E_\rho) \).

A metric on \( TX \) is a Riemannian metric. Now, the length-squared function for the dual metric

\[
\| \rho \|^2: T^*_p X \to \mathbb{R}
\]

is a homogeneous polynomial of degree two.
A generalized Dirac formula in a bundle $E$ (complex) on a fibre-wise diffeomorphism $\varphi \in \text{Diff}^1(X; E)$ such that

$$(i) \quad (\varphi^*, \varphi^*) = 1 \cdot 1^2 \times \text{Id}$$

for a metric on $\varphi^*$ (but $E$!)

What does this mean? Consider, for $f \in X$,

its square

$$(\varphi^* f)^2 = f \varphi^* f$$

This defines a linear map

$$T \varphi^* f : T^* X \rightarrow T \varphi^* f \in \text{Hom}(E)$$

Let's see its effect at (1). If $S_1, S_2 \in X$

then

$$\varphi^* (d(S_1 + S_2))^2 = |S_1 + S_2|^2$$
\[ \sum_{k=0}^{N} V^k \left/ \left( e_{10} e_5 + e_{01} e_5 - 2 e_{10} \right) \right. \]

As a real space it is complete \( \mathbb{R}^V \).