QUESTIONS 18.102 FINAL, SPRING 2013

I expect to choose 6 of these questions on the final. Note that some are rather straightforward and some are less so, I will choose some of each. Before the day of the exam, you can ask me for hints.

All Hilbert spaces should be taken to be separable and non-trivial, i.e. containing an element other than 0.

Problem 1

Let $H_i, i = 1, 2$ be two Hilbert spaces with inner products $(\cdot, \cdot)_i$ and suppose that $I : H_1 \rightarrow H_2$ is a continuous linear map between them. Suppose that

1. the range of $I$ is dense in $H_2$
2. $I$ is injective.
3. if $v_n \rightharpoonup v$ (converges weakly) in $H_1$ then $I(v_n) \rightarrow I(v)$ in $H_2$.

Show that

1. there is a continuous linear map $Q : H_2 \rightarrow H_1$ such that $(u, I(f))_2 = (Qu, f)_1 \forall f \in H_1$.
2. as a map from $H_1$ to itself, $Q \circ I$ is bounded and self-adjoint.
3. as a map on $H_1$, $Q \circ I$ is compact.
4. as a map on $H_2$, $I \circ Q$ has the same eigenvalues as $Q \circ I$ on $H_1$.

Problem 2

Let $A_j \subset \mathbb{R}$ be a sequence of subsets with the property that the characteristic function, $\chi_j$ of $A_j$, is integrable for each $j$. Show that the characteristic function of $\mathbb{R} \setminus A$, where $A = \bigcup_j A_j$ is locally integrable.

Problem 3

If $H$ is a Hilbert space let $l^2(H)$ be the space of sequences $h : \mathbb{N} \rightarrow H$ such that $\sum_j ||u(j)||^2_H < \infty$. Show that this is a Hilbert space and that there is a bounded linear bijection $l^2(H) \rightarrow H$ if and only if $H$ is not finite dimensional.

Problem 4

Let $A$ be a Hilbert-Schmidt operator on a separable Hilbert space $H$, which means that for some orthonormal basis $\{e_i\}$

1. $\sum_i ||Ae_i||^2 < \infty$.

Using Bessel’s identity to expand $||Ae_i||^2$ with respect to another orthonormal basis $\{f_j\}$ show that $\sum_j ||A^*f_j||^2 = \sum_i ||Ae_i||^2$. Conclude that the sum in (1) is independent of the orthonormal basis used to define it and that the Hilbert-Schmidt operators form a Hilbert space.
Problem 5

Let $a : [0, 2\pi] \rightarrow L^2(0, 2\pi)$ be a continuous map. Show that

$$Af(x) = \int_0^{2\pi} a(x)f, \ f \in L^2(0, 2\pi)$$

defines a compact operator from $L^2(0, 2\pi)$ to $C([0, 2\pi])$ (i.e. the image of a bounded set is equicontinuous). Conversely show that if $A : L^2(0, 2\pi) \rightarrow C([0, 2\pi])$ is given as a compact (in particular bounded) operator for the supremum norm on $C([0, 2\pi])$ then there exists such a map $a$.

Problem 6

Let $u_n : [0, 2\pi] \rightarrow \mathbb{C}$ be a sequence of continuously differentiable functions which is uniformly bounded, with bounded derivatives i.e. $\sup_x \sup_{[0,2\pi]} |u_n(x)| < \infty$ and $\sup_x \sup_{[0,2\pi]} |u_n'(x)| < \infty$. Show that $u_n$ has a subsequence which converges in $L^2([0, 2\pi])$.

Problem 7

Suppose that $f \in C^1(0, 2\pi)$ is such that the constants $c_k = \int_{(0, 2\pi)} f(x)e^{-ikx}$, $k \in \mathbb{Z}$, satisfy $\sum_{k \in \mathbb{Z}} |c_k|^2 < \infty$. Show that $f \in L^2(0, 2\pi)$.

Problem 8

Carefully justify each step in the following proof of the uniform boundedness principle (see Alan D. Sokal, ‘A really simple elementary proof of the uniform boundedness theorem’, Arxiv math:1005.1585).

Theorem: If a sequence, or more generally a collection $\mathcal{T}$, of bounded operators $T : V \rightarrow W$, where $V$ is a Banach space and $W$ is a normed space, is such that for each $v \in V$, $\sup_{T \in \mathcal{T}} \|T(v)\|_W < \infty$ then $\sup_{T \in \mathcal{T}} \|T\| < \infty$.

Proof:

1. Suppose to the contrary that $\sup_{T \in \mathcal{T}} \|T\| = \infty$.
2. Choose a sequence $T_k$ in $\mathcal{T}$ such that $\|T_k\| \geq 4^k$.
3. Observe that for a bounded operator, $S$, between two normed spaces

$$\max[\|S(v + v')\|, \|S(v - v')\|] \geq \frac{1}{2}[\|S(v + v')\| + \|S(v - v')\|] \geq \|S'\|$$

4. Deduce that

$$\sup_{w \in B(v, r)} \|Su\| \geq \|S\|r \ \forall \ v \in V \text{ and } r > 0.$$ 

5. Set $v_0 = 0$ and note that for $k \geq 1$, proceeding inductively, points $v_k \in V$ can be chosen such that $\|v_k - v_{k-1}\| \leq 3^{-k}$ and $\|T_k v_k\| \geq \frac{4}{3} 3^{-k} \|T_k\|$.

6. The sequence $\{v_k\}_{k=1}^\infty$ is Cauchy, hence converges to some $v \in V$.

7. Since $\|v - v_k\| \leq \frac{3}{2} 3^{-k}$, $\|T_k v\| \geq \frac{1}{2} 3^{-k} \|T_k\| \geq \frac{1}{2} \frac{4}{3} 3^{-k} \rightarrow \infty$.

8. Hence $\sup_{T \in \mathcal{T}} \|T\| < \infty$. 

**Problem 9**

Let \( B_n \) be a sequence of bounded linear operators on a Hilbert space \( H \) such that for each \( u \) and \( v \in H \) the sequence \((B_n u, v)\) converges in \( \mathbb{C} \). Show that there is a uniquely defined bounded operator \( B \) on \( H \) such that
\[
(Bu, v) = \lim_{n \to \infty} (B_n u, v) \quad \forall \ u, v \in H.
\]

**Problem 10**

Let \( T : H_1 \to H_2 \) be a continuous linear map between two Hilbert spaces and suppose that \( T \) is both surjective and injective.

1. Let \( A_2 \in \mathcal{K}(H_2) \) be a compact linear operator on \( H_2 \), show that there is a compact linear operator \( A_1 \in \mathcal{K}(H_1) \) such that
\[
A_2 T = TA_1.
\]
2. If \( A_2 \) is self-adjoint (as well as being compact) and \( H_1 \) is infinite dimensional, show that \( A_1 \) has an infinite number of linearly independent eigenvectors.

**Problem 11**

Suppose \( P \subset H \) is a closed linear subspace of a Hilbert space, with \( \pi_P : H \to P \) the orthogonal projection onto \( P \). If \( H \) is separable and \( A \) is a compact self-adjoint operator on \( H \), show that there is a complete orthonormal basis of \( H \) each element of which satisfies \( \pi_P A \pi_P e_i = t_i e_i \) for some \( t_i \in \mathbb{R} \).

**Problem 12**

Let \( e_j = c_j C^j e^{-x^2/2} \), \( c_j > 0 \), where \( j = 1, 2, \ldots \), and \( C = -\frac{d}{dx} + x \) is the creation operator, be the orthonormal basis of \( L^2(\mathbb{R}) \) consisting of the eigenfunctions of the harmonic oscillator discussed in class. You may assume completeness in \( L^2(\mathbb{R}) \) and use the facts established in class that
\[
-\frac{d^2}{dx^2} e_j + x^2 e_j = (2j + 1) e_j,
\]
and that \( e_j = p_j(x) e_0 \) for a polynomial of degree \( j \). Compute \( C e_j \) and \( A e_j \) in terms of the basis and hence arrive at a formula for \( \frac{de_j}{dx} \). Use this to show that the sequence \( j^{-\frac{1}{2}} \frac{de_j}{dx} \) is bounded in \( L^2(\mathbb{R}) \). Conclude that if
\[
H^1_{iso} = \{ u \in L^2(\mathbb{R}); \sum_{j \geq 1} j |(u, e_j)|^2 < \infty \}
\]
then there is a uniquely defined operator \( D : H^1_{iso} \to L^2(\mathbb{R}) \) such that \( D e_j = \frac{de_j}{dx} \) for each \( j \).

**Problem 13**

Let \( A \) be a compact self-adjoint operator on a separable Hilbert space and suppose that for any orthonormal basis
\[
\sum_i |(A e_i, e_i)| < \infty.
\]
Show that the eigenvalues of \( A \), if infinite in number, form a sequence in \( l^1 \).
Problem 14

Consider the subspace $H \subset C[0, 2\pi]$ consisting of those continuous functions on $[0, 2\pi]$ which satisfy

$$u(x) = \int_0^x U(x), \quad \forall x \in [0, 2\pi]$$

for some $U \in L^2(0, 2\pi)$ (depending on $u$ of course). Show that the function $U$ is determined by $u$ (given that it exists), that

$$\|u\|_H^2 = \int_{(0, 2\pi)} |U|^2$$

turns $H$ into a Hilbert space.

Problem 15

Let $e_j = c_j e^{-x^2/2}, \ c_j > 0$, where $C = -\frac{d}{dx} + x$ is the creation operator, be the orthonormal basis of $L^2(\mathbb{R})$ consisting of the eigenfunctions of the harmonic oscillator discussed in class. Define an operator on $L^2(\mathbb{R})$ by

$$A u = \sum_{j=0}^{\infty} (2j + 1)^{-\frac{1}{2}} (u, e_j)_{L^2} e_j.$$  

1. Show that $A$ is compact as an operator on $L^2(\mathbb{R})$.

2. Suppose that $V \in C_0^\infty(\mathbb{R})$ is a bounded, real-valued, continuous function on $\mathbb{R}$. Show that $L^2(\mathbb{R})$ has an orthonormal basis consisting of eigenfunctions of $K = AVA$, where $V$ is acting by multiplication on $L^2(\mathbb{R})$.

Problem 16

Suppose that $f \in L^1(0, 2\pi)$ is such that the constants

$$c_k = \int_{(0,2\pi)} f(x) e^{-ikx}, \quad k \in \mathbb{Z},$$

satisfy

$$\sum_{k \in \mathbb{Z}} |c_k|^2 < \infty.$$ 

Show that $f \in L^2(0, 2\pi)$.

Problem 17

Consider the space of those complex-valued functions on $[0, 1]$ for which there is a constant $C$ (depending on the function) such that

$$|u(x) - u(y)| \leq C|x - y|^\frac{1}{2} \quad \forall x, y \in [0, 1].$$

Show that this is a Banach space with norm

$$\|u\|_2 = \sup_{[0, 1]} |u(x)| + \inf_{(5) \text{ holds}} C.$$
Problem 18

Let $B : L^2(\mathbb{R}) \times L^2(\mathbb{R}) \rightarrow \mathbb{C}$ be a bilinear form (meaning it is linear in each factor when the other is held fixed) such that there is a constant $C > 0$ and

$$|B(u, v)| \leq C\|u\|_L^2\|v\|_L^2 \quad \forall \ u, v \in L^2(\mathbb{R}).$$

Show that there is a bounded linear operator $T : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ such that

$$\int_{\mathbb{R}} T(u)(x)v(x) = B(u, v) \quad \forall \ u, v \in L^2(\mathbb{R}).$$