Try each of the questions; they will be given equal value. You may use theorems from class, or the book, provided you can recall them correctly!

**Problem 1**

Consider the set \( S \) defined as follows. The elements of \( S \) are sequences, \( \{s_n\}_{n=1}^{\infty} \) with all entries either 1 or 2 and with the additional property that every 2 is followed by a 1. Said more precisely, for every \( n, s_n = 1 \) or \( s_n = 2 \) and if \( s_n = 2 \) then \( s_{n+1} = 1 \). Say why precisely one of the following is true

(a) \( S \) is finite
(b) \( S \) is countably infinite
(c) \( S \) is uncountably infinite

and then decide which one is true and prove it.

**Problem 2**

Consider the metric space \( M = [0,1] = \{x \in \mathbb{R}; 0 \leq x \leq 1\} \) with the usual metric, \( d(x,y) = |x - y| \). Is the set \( A = [0, \frac{1}{2}) = \{x \in \mathbb{R}; 0 \leq x < \frac{1}{2}\} \) open as a subset of \( M \)? What is the closure of \( A \) as a subset of \( M \)? Is \( A \) compact? Is the closure of \( A \) compact? In each case justify your answer.

**Problem 3**

Let \( M \) be a compact metric space. Suppose \( A \subset M \) is not compact. Show, directly from the definition or using a theorem proved in class, that \( A \) is not closed.

**Problem 4**

Recall that a set \( S \) in a metric space \( M \) is connected if any separated decomposition of it, \( S = A \cup B \) where \( A \cap B = \emptyset = A \cap \overline{B} \), is ‘trivial’ in the sense that either \( A \) or \( B \) is empty. Show that the whole metric space \( M \) is connected if and only if the only subsets \( A \subset M \) of it which are both open and closed are the ‘trivial’ cases \( A = \emptyset \) and \( A = M \).

**Another possible Test**

**Problem 1**

Show that the set \( \{0\} \cup \{1/n; n \in \mathbb{N}\} \) is compact as a subset of the metric space \( \mathbb{Q} \), the rational numbers, with the usual metric \( d(x,y) = |x - y| \).

**Problem 2**

Let \( X \) be a set with the discrete metric, \( d(x,x) = 0 \) and \( d(x,y) = 1 \) if \( x \neq y \). Show that every function \( f : X \rightarrow \mathbb{C} \) is continuous.
Problem 3
Consider the metric $d(x, y) = d_1((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|$ on $\mathbb{R}^2$.

1) Show that if $d$ is the usual metric on $\mathbb{R}^2$ then
   $$d(x, y) \leq d_1(x, y) \leq 2d(x, y) \forall x, y \in \mathbb{R}^2.$$  
2) Show that the open sets relative to $d_1$ are the same as those relative to $d$.

Problem 4
Suppose that $X$ is a metric space and $A \subset X$ is an open set which is compact and is neither empty nor equal to $X$. Show that $X$ is not connected.