18.100B, Fall 2002, Homework 4, Solutions

Was due in 2-251, by Noon, Tuesday October 1. Rudin:

(1) Chapter 2, Problem 22
Let \( Q^k \subset \mathbb{R}^k \) be the subset of points with rational coefficients. This is countable, as the Cartesian product of a finite number of countable sets. Suppose that \( x = (x_1, \ldots, x_k) \in \mathbb{R}^k \). By the density of the rationals in the real numbers, given \( \epsilon > 0 \) there exists \( y_i \in \mathbb{Q}^k \) such that \(|x_i - y_i| < \epsilon/k \), \( i = 1, \ldots, k \). Thus if \( y = (y_1, y_2, \ldots, y_k) \) then

\[
|x - y| \leq \sqrt{k} \max_{i=1}^{k} |x_i - y_i| < \epsilon
\]

shows the density of \( Q^k \) in \( \mathbb{R}^k \). Thus \( \mathbb{R}^k \) is separable.

(2) Chapter 2, Problem 23
Given a separable metric space \( X \), let \( Y \subset X \) be a countable dense subset. The product \( A = Y \times \mathbb{Q} \) is countable. Let \( \{U_a\}, a \in A \), be the collection of open balls with center from \( Y \) and rational radius. If \( V \subset X \) is open then for each point \( p \in V \) there exists \( r > 0 \) such that \( B(p, r) \subset V \). By the density of \( Q \) in \( X \) there exists \( y \in Q \) such that \( p \in B(y, r/2) \). Moreover there exists \( q \in Q \) with \( r/2 < q < r \). Then \( x \in B(y, q) \). Thus each point of \( V \) is in an element of one of the \( U_a \)'s which is contained in \( V \), so

\[
V = \bigcup_{U_a \subset V} U_a.
\]

It follows that the \( \{U_a\}_{a \in A} \) form a base of \( X \) (actually now more usually called an open basis).

(3) Chapter 2, Problem 24
By assumption \( X \) is a metric space in which every infinite set has a limit point.

For each positive integer \( n \) choose points \( x_1(n), x_2(n), \ldots \) successively with the property that \( d(x_j(n), x_k(n)) \geq 1/n \) for \( k < j \). After a finite number of steps no further choice is possible. Indeed, if there were an infinite set of points \( E \) satisfying \( d(x, x') \geq 1/n \) for all \( x \neq x' \) in \( E \) then \( E \) could have no limit point – since a limit point \( q \in X \) would have to satisfy \( d(q, p_i) < 1/2n \) for an infinite number of (different) \( p_i \in E \) and this would imply that \( d(p_1, p_2) \leq d(p_1, q) + d(q, p_2) < 1/n \) which is a contradiction. Let \( Y \subset X \) be the countable subset, as a countable union of finite sets, consisting of all the \( x_j(n) \), for all \( n \). Then \( Y \) is dense in \( X \). To see this, given \( p \in X \) and \( \epsilon > 0 \) choose \( n > 1/\epsilon \). If \( p = x_j(n) \) for some \( j \) then it is in \( Y \). If not then for some \( j \), \( d(p, x_j(n)) < 1/n \); otherwise it would be possible to choose another \( x_j(n) \) contradicting the fact that we have chosen as many as possible. Then \( d(p, q) < \epsilon \) for some \( q \in Y \) which is therefore dense and \( X \) is therefore separable.

(4) Chapter 2, Problem 26
By assumption, \( X \) is a metric space in which every infinite subset has a limit point. By the problems above it is separable, and hence has a countable open basis, \( \{U_i\} \). Let \( \{V_a\}_{a \in A} \) be an arbitrary open cover of \( X \). Each \( V_a \) is a union of \( U_j \)'s by the definition of an open basis. For each \( j \) such that \( U_j \) is in one of these unions, choose a \( V_{a_j} \) which contains it. Then for every \( b \in A \), \( V_b \) must be contained in a union of the \( U_{a_j} \)'s, hence in the
union of the \( V_{a_i} \)'s which therefore form a countable subcover of the original open cover \( V_a \). Consider the successive open sets

\[
\bigcup_{i=1}^{N} V_{a_i}.
\]

If one of these contains \( X \) then we have found a finite subcover of the \( V_a \)'s. So, suppose to the contrary that

\[
F_N = X \setminus \bigcup_{i=1}^{N} V_{a_i} \neq \emptyset \quad \forall \ N.
\]

The \( F_N \)'s are decreasing as \( N \) increases. Let \( E \subset X \) be a set which contains one point from each \( F_N \). It must be an infinite set, since otherwise some fixed point would be in \( F_N \) for arbitrary large, hence all, \( N \) but

\[
\bigcap_{N \in \mathbb{N}} F_N = \emptyset
\]

Since together all the \( V_{a_i} \) do cover \( X \). By the assumed property of \( X \), \( E \) must have a limit point \( p \). For each \( N \), all but finitely many points of \( E \) lie in \( F_N \), so \( p \) must be a limit point of \( F_N \) for all \( N \), but each \( F_N \) is closed so this would mean \( p \in F_N \) for all \( N \), contradicting (1).