PROBLEM SET 2, 18.157
SOLUTIONS TO REWORDED QUESTIONS!

This was a bit of a disaster, my fault I think – in 1) I should have given you a pointer, 2) was grossly misstated 3) I did not ask the question directly enough

(1) Recall that the wavefront set of a distribution can be written as the intersections of the characteristic varieties of pseudodifferential operators which map it to a smooth function. Maybe using differential operators show that the wavefront set of the distribution

\[ u(x) = \begin{cases} 1 & |x| \leq 1 \\ 0 & |x| > 1 \end{cases} \in \mathcal{S}(\mathbb{R}^n) \]

is contained in a smooth conic manifold of dimension \( n \) (which is in fact a Lagrangian submanifold of \( T^*\mathbb{R}^n \)).

Solution: Using coordinate independence to replace the sphere by a plane works – although I have not proved coordinate independence for the wavefront set yet – then one can use the Fourier transform.

A direct way is to observe that the singular support is \( |x| = 1 \) so \( \text{WF}(u) \) must lie above it. Now, the definition is

\[ \text{WF}(u) = \bigcap \{ \Sigma(A); Au \in C^\infty \}. \]

We can find some differential operators, namely the generators of the rotation group, which annihilate \( u \):

\[ (x_i D_j - x_j D_i)u = 0. \]

So \( x_i \xi_j - x_j \xi_i = 0 \) on \( \text{WF}(u) \). Fixing a point, \( x = (1, 0, \ldots, 0) \) we see that this means \( \xi_j = 0 \) for \( j \neq 1 \) and in general that \( (x, \xi) \notin \text{WF}(u) \) if \( \xi \neq a \pm x, a > 0 \). So as a conic set the wavefront set is contained in the conormal variety to the sphere

\[ \text{WF} \subset N^*S \subset T^*\mathbb{R}^n, \]

consisting of the (non-vanishing) multiples of \( d|x|^2 \) at \( |x| = 1 \). This is clearly a smooth manifold, since it is just \( \mathbb{S}^{n-1} \times (\mathbb{R} \setminus \{0\}) \).

What about the converse, that the wavefront set is no smaller than this?
(2) (Reminder) Show that if \( f \in C_c^{-\infty}(\mathbb{R}^n) \) and \( \lambda \notin [0, \infty) \) then
\[ u \in S'(\mathbb{R}^n), \quad (\Delta - \lambda)u = f \implies u \in C_c^{-\infty}(\mathbb{R}^n) + S(\mathbb{R}^n). \]

Solution: I really messed this up. By elliptic regularity, the singular support of \( u \) must be compact, contained in that of \( f \), and if we cut off
\[ u = u_1 + u_2, \quad u_1 = \phi u, \quad u_2 = (1 - \phi)u \]
where \( \phi \in C_c^\infty(\mathbb{R}^n) \) is identically equal to 1 in a neighbourhood of the support of \( f \) then \((\Delta - \lambda)u_2 \in C_c^\infty(\mathbb{R}^n)\). We can simply look at the Fourier transform
\[ \hat{u}_2(\xi) = \hat{f}/(|\xi|^2 - \lambda) \in S(\mathbb{R}^n) \implies u_2 \in S(\mathbb{R}^n). \]

(3) Consider a metric on \( \mathbb{R}^n \) which is a compactly supported perturbation of the Euclidean metric
\[ g_{ij}(x) = \delta_{ij} \text{ in } |x| > R, \quad \sum_{ij} g_{ij}(x)\xi^i\xi^j \geq c|\xi|^2, \quad c > 0. \]

Show, for \( \lambda \notin [0, \infty) \) that
\[ u \in S'(\mathbb{R}^n), \quad (\Delta_g - \lambda)u \in \langle x \rangle^k L^2(\mathbb{R}^n) \implies u \in \langle x \rangle^k H^2(\mathbb{R}^n). \]

Hint: Use the previous question and then elliptic regularity.
Solution: I did not work this out carefully ... Same set up as previous question. We know that \( \Delta_g - \lambda \) has a bounded inverse on \( L^2(\mathbb{R}^n) \) for \( \lambda \notin [0, \infty) \). We would like to prove that it is a ‘standard’ pseudodifferential operator. What can you deduce from the identity
\[ (\Delta - \lambda)(\Delta_g - \lambda)^{-1} + P(\Delta_g - \lambda)^{-1} = \text{Id} \]
where \( \Delta \) is the flat Laplacian? What about using the adjoint of this as well. Here \( P = \Delta_g - \Delta \) has compactly-supported coefficients.
Solution: What follows from (5) is that
\[ (\Delta_g - \lambda)^{-1} = - (\Delta - \lambda)^{-1}P(\Delta_g - \lambda)^{-1} + (\Delta - \lambda)^{-1}. \]

There is a similar identity for composition the other way:
\[ (\Delta_g - \lambda)^{-1} = - (\Delta_g - \lambda)^{-1}P(\Delta - \lambda)^{-1} + (\Delta - \lambda)^{-1}. \]

Inserting the second one in the first gives
\[ (\Delta_g - \lambda)^{-1} = (\Delta - \lambda)^{-1}P(\Delta_g - \lambda)^{-1}P(\Delta - \lambda)^{-1} - (\Delta_g - \lambda)^{-1}P(\Delta - \lambda)^{-1} + (\Delta - \lambda)^{-1} \]

Now, we do know from the parametrix construction that
\[ (\Delta_g - \lambda)^{-1} = Q + R, \quad Q \in \Psi^{-1}_\infty(\mathbb{R}^n), \]
where $R$ is a smoothing operator, which maps $H^k(\mathbb{R}^n)$ to $H^\infty(\mathbb{R}^n)$ for all $k \in \mathbb{R}$ and with adjoint which does the same – what we don’t know is that $R \in \Psi^{-\infty}(\mathbb{R}^n)$.

In fact it follows from (6) that indeed

\[
(\Delta_g - \lambda)^{-1} \in \Psi^{-2}(\mathbb{R}^n)
\]

since the second two terms are in the space and the middle part of the first term $P(\Delta_g - \lambda)^{-1}P$ is the sum of a element of this space and a compactly supported smoothing operator – which is also in the space.

(4) Show that if $A \in \Psi^m(U)$, the pseudodifferential operators on an open subset of $\mathbb{R}^n$ is properly supported, meaning in terms of the Schwartz’ kernel the two projections from $U \times U$ restrict to proper maps

\[
\pi_L, \pi_R : \text{supp}(A) \longrightarrow U
\]

then

\[
A : H^s_{\text{loc}}(U) \longrightarrow H^{s-m}_{\text{loc}}(U) \; \forall \; s.
\]

Hint: Freely use uniqueness of Schwartz’ kernels on open sets, maybe prove the adjoint preserves the spaces of compact support?

Solution: We know that $A^* \in \Psi^m(U)$ is a continuous linear map from $H^{-s+m}_{\text{c}}(U)$ to $H^{-s}_{\text{c}}(U)$. The properness of the support of $A$, and hence of $A^*$ means that

\[
A^* : H^{-s+m}_{\text{c}}(U) \longleftrightarrow H^{-s}_{\text{c}}(U)
\]

and then by duality $A$ is a map as suggested.