PROBLEM SET 6 FOR 18.102
DUE 7AM SATURDAY MARCH 28, 2015.

Problem 6.1

Recall the space \( h^{2,1} \) (discussed in an earlier problem set) consisting of the complex valued sequences \( c_i \) such that
\[
\|c\|^2 = \sum_i (1 + |i|^2)|c_i|^2 < \infty.
\]
Show that the unit ball in this space, considered as a subset of \( l^2 \), has compact closure.

Hint: You may use the criterion for compactness of a set in a separable Hilbert space which was proved in lectures and is in the notes.

Problem 6.2

Define the space \( L^2(0,1) \) as consisting of those elements of \( L^2(\mathbb{R}) \) which vanish outside \((0,1]\) and show that the quotient \( L^2(0,1) = L^2(0,1)/N(0,1) \) by the null functions in \( L^2(0,1) \) is a Hilbert space.

Remark: This is indeed easy, but make sure you do it properly (for instance identify \( L^2(0,1) \) with a closed subspace of \( L^2(\mathbb{R}) \) by treating the null functions properly).

Problem 6.3

Identify \( C[0,1] \), the space of continuous functions on the closed interval, as a subspace of \( L^2(0,1) \). For each \( n \in \mathbb{N} \) let \( F_n \subset L^2(0,1) \) be the subspace of functions which are constant on each interval \(((m-1)/n, m/n] \) for \( m = 1, \ldots, n \) Show that if \( f \in C[0,1] \) there exists \( g_n \in F_n \) such that
\[
\delta_n = \sup_{|t-s| \leq 1/n} |f(t) - f(s)| \implies \|f - g_n\|_{L^2} \leq \delta_n.
\]

Problem 6.4

Show that a bounded and equicontinuous subset of \( C[0,1] \) has compact closure in \( L^2(0,1) \). Note that equicontinuity means 'uniform equicontinuity' so for each \( \epsilon > 0 \) there exists \( \delta > 0 \) such that \( |x - y| < \delta \) implies \( |f(x) - f(y)| < \epsilon \) for all elements \( f \) of the set.

Hint: Use the approximation by the finite dimensional spaces \( F_n \).

Problem 6.5

If \( K \in C([0,1] \times [0,1]) \) is a continuous function of two variables, show that
\[
Af(x) = \int K(x,y)f(y)
\]
defines a compact linear operator on \( L^2(0,1) \).
Hint: Show that $A$ defines a bounded linear map from $L^2(0,1)$ to $C[0,1]$ and that the image of the unit ball is *equicontinuous* using the uniform continuity of $K$.

Problem 6.6 – extra

Show that a closed and bounded subset of $L^2(\mathbb{R})$ is compact if and only if it is ‘uniformly equicontinuous in the mean’ and ‘uniformly small at infinity’ so that for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$\int_{-\delta}^{1/\delta} |f|^2 < \epsilon^2 \quad \text{and} \quad |t| < \delta \implies \int |f(x) - f(x-t)|^2 < \epsilon^2$$

for all elements of the set.

Problem 6.7 – extra

Consider the space of continuous functions on $\mathbb{R}$ vanishing outside $(0,1)$ which are of the form

$$u(x) = \int_0^x v, \quad v \in L^2(0,1).$$

Show that these form a Hilbert space and that the unity ball of this space has compact closure in $L^2(0,1)$. 