SHORT SOLUTIONS FOR 18.102 FINAL EXAM, SPRING 2015

PROBLEM 1

Consider the subspace $H \subset C[0, 2\pi]$ consisting of those continuous functions on $[0, 2\pi]$ which satisfy

(1) $u(x) = \int_0^x U, \ \forall \ x \in [0, 2\pi]$ for some $U \in L^2(0, 2\pi)$ (depending on $u$ of course). Show that the function $U$ is determined by $u$ (given that it exists) and that

(2) $\|u\|_H^2 = \int_{(0, 2\pi)} |U|^2$

turns $H$ into a Hilbert space.

Solution: If $U \in L^2([0, 2\pi])$ then the integral (1) defines a continuous function since

$$|u(x) - u(y)| \leq \int_y^x |U| \leq |x - y|^{1/2} \|U\|_{L^2}, \ \sup |u| \leq (2\pi)^{1/2} \|U\|_{L^2}$$

so in fact $I : L^2[0, 2\pi] \rightarrow C([0, 2\pi])$ is a bounded linear map. To say that $U$, if it exists, is determined by $u$ is to say that this map in injective. The vanishing of $u$ means precisely that $\langle \chi_{[0,x]}, U \rangle_{L^2} = 0$. Taking linear combination, this means that $U$ is orthogonal to all step functions. However the step functions are dense in $C([0, 2\pi])$ in the supremum norm and hence in $L^2[0, 2\pi]$, so this imples $U = 0$ in $L^2$. Since $I$ is injective, it is a bijection onto its range, $H$ and this gives a bijection to $L^2[0, 2\pi]$, making $H$ into a Hilbert space.

Other arguments that work include computing the Fourier coefficients of $U$ to shows that they are determined by $u$. In general a measurable set (where $U > 0$ for instance) does not contain a close measurable set of positive measure, so that sort of approach is hard.

PROBLEM 2

Consider the space of those complex-valued functions on $[0, 1]$ for which there is a constant $C \geq 0$ (depending on the function) such that

(3) $|u(x) - u(y)| \leq C|x - y|^{1/2} \ \forall \ x, y \in [0, 1]$.

Show that this is a Banach space with norm

(4) $\|u\|_\frac{1}{2} = \sup_{[0,1]} |u(x)| + \inf_{(3) \ \text{holds}} C$. 

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Solution: These are the Hölder-$\frac{1}{2}$ functions, $C^{\frac{1}{2}}[0, 1]$. If (3) holds for some constant $C \geq 0$ then
\[ \|u\|' = \sup_{x \neq y \in [0, 1]} \frac{|u(x) - u(y)|}{|x - y|^{\frac{1}{2}}} < \infty \]
is the smallest such constant and the putative norm is
\[ \|u\|_{\frac{1}{2}} = \sup_{[0, 1]} |u(x)| + \|u\|'. \]
I expected you to quickly check that this is a norm and that the space of functions $C^{\frac{1}{2}}[0, 1]$ is linear. The inequality (3) implies that the elements of $C^{\frac{1}{2}}$ are continuous and if $u_n$ is a Cauchy sequence it follows that it is Cauchy with respect to the supremum norm, $\|u\|_{\infty} \leq \|u\|_{\frac{1}{2}}$ by definition. Since this space is complete, $u_n \to u$ uniformly with $u : [0, 1] \to \mathbb{C}$ continuous. A Cauchy sequence is bounded in norm so
\[ |u_n(x) - u_n(y)| \leq C|x - y|^\frac{1}{2} \]
with $C$ independent of $n$. Passing to the limit $n \to \infty$ shows that $u \in C^{\frac{1}{2}}$. The Cauchy condition itself implies that given $\epsilon > 0$ there exists $N$ such that
\[ |(u_n(x) - u_m(x)) - (u_n(y) - u_m(y))| \leq C\epsilon|x - y|^\frac{1}{2} \quad \forall \, n, m > N. \]
Taking $m \to \infty$ and using the convergence in supremum norm it follows that $\|u - u_n\|_{\frac{1}{2}} \to 0$.
Generally well done.

**Problem 3**

Let $A_j \subset \mathbb{R}$ be a sequence of subsets with the property that the characteristic function, $\chi_j$ of $A_j$, is integrable for each $j$. Show that the characteristic function of $\mathbb{R} \setminus A$, where $A = \bigcup_j A_j$ is locally integrable.

Solution: Since for each $j$, $\chi_j \in L^1(\mathbb{R})$ are real functions it follows that $\chi_{[k]}$, the characteristic function of $\bigcup_{j \leq k} A_j$ is in $L^1(\mathbb{R})$ as the supremum of a finite number of $L^1$ functions and so is $\chi_{[-R,R]}\chi_{[k]}$ for each $R > 0$. The $L^1$ integral of this increasing sequence if bounded by $4R$ so by Monotone Convergence, $\chi_{[-R,R]}\chi_A \in L^1(\mathbb{R})$ where $\chi_A$ is the characteristic function of $A = \bigcup_j A_j$. The difference $\chi_{[-R,R]}(1 - \chi_{[\infty]})$ is therefore also integrable and this is $\chi_{[-R,R]}\chi_B$ where $B = \mathbb{R} \setminus A$, so $\chi_B$ is locally integrable.

**Problem 4**

Let $A$ be a Hilbert-Schmidt operator on a separable Hilbert space $H$, which means that for some orthonormal basis $\{e_i\}$
\[ \|A\|^2_{HS} = \sum_i \|Ae_i\|^2 < \infty. \]
Using Bessel’s identity to expand $\|Ae_i\|^2$ with respect to another orthonormal basis $\{f_j\}$ show that $\sum_j \|A^* f_j\|^2 = \sum_i \|Ae_i\|^2$. Conclude that the sum in (5) is independent of the orthonormal basis used to define it and that the Hilbert-Schmidt operators form a Hilbert space.

Solution: Everyone got the proof that the Hilbert-Schmidt norm is independent of the onb. I expected you to quickly check linearity and the norm properties.

Taking a unit vector $u$ and an orthonormal basis $e_i$ and orthonormalizing the sequence $u, e_1, \ldots$, gives an orthonormal sequence with first element $u$. Thus

$$\|Au\| \leq \|A\|_{HS} \implies \|A\| \leq \|A\|_{HS}$$

So, if $A_n$ is Cauchy with respect to the Hilbert-Schmidt norm it is Cauchy in the norm on $\mathcal{B}$, which is complete, so $A_n \to A$ in norm. A Cauchy sequence is bounded in norm so for any finite $M$ it follows that

$$\sum_{i < M} \|A_n e_i\|^2 \leq \sup \|A_n\|_{HS} \leq C < \infty.$$ 

Passing to the limit as $n \to \infty$ using norm convergence and then letting $M \to \infty$ it follows that $A$ is Hilbert-Schmidt and then the Cauchy condition shows that given $\epsilon > 0$ there exists $N$ such that $n, m > N$ implies

$$\sum_{i < M} \|A_n e_i - A_m e_i\|^2 \leq \epsilon^2 \forall M.$$ 

Taking $m \to \infty$ then $M \to \infty$ it follows that $A_n \to A$ in the Hilbert-Schmidt norm.

**Problem 5**

Let $A$ be a compact self-adjoint operator on a separable Hilbert space and suppose that for every orthonormal basis

$$\sum_i |(Ae_i, e_i)| < \infty. \tag{6}$$

Show that the eigenvalues of $A$, if infinite in number, form a sequence in $l^1$. Solution:

Every compact self-adjoint operator has an orthonormal basis of eigenvectors so if the eigenvalues are listed with multiplicity then

$$\sum_i |\lambda_i| = \sum_i |(Ae_i, e_i)| < \infty$$

from (6). If the eigenvalues are listed without multiplicity, the sum is smaller so still in $l^1$. [Either interpretation is acceptable.]

**Problem 6**

For $u \in L^2(0, 1)$ show that

$$Iu(x) = \int_0^x u(t)dt, \ x \in (0, 1)$$
is a bounded linear operator on $L^2(0,1)$. If $V \in C([0,1])$, is real-valued and $V \geq 0$, show that there is a bounded linear operator $B$ on $L^2(0,1)$ such that
\[ (7) \quad B^2 u = u + I^* M_V I u \quad \forall u \in L^2(0,1) \]
where $M_V$ denotes multiplication by $V$.

Solution: $Iu$ is continuous if $u \in L^2(0,1)$ (see Problem 1 ...) since
\[ |u(x) - u(y)| \leq \int_x^y |u| \leq |x - y|^\frac{1}{2} \|u\|_{L^2} \]
by Cauchy-Schwartz. By the linearity of the integral, this is a linear map from $L^2(0,1)$ to $C([0,1])$ and
\[ \|Iu\|_{L^2} \leq \sup |u| \leq \|u\|_{L^2} \]
so it is bounded on $L^2$. The image of the unit ball in $L^2(0,1)$ is a uniformly bounded and equicontinuous set in $C(0,1)$ so has compact closure by Arscoli-Arzela. The image under the inclusion into $L^2(0,1)$ is therefore also precompact and hence $I$ is a compact operator.

Multiplication by a continuous function $V \geq 0$ gives a bounded and self-adjoint operator on $L^2(0,1)$,
\[ \|M_V\| \leq \sup V, \quad \langle M_V u, v \rangle = \int V u \overline{v} = \langle u, M_V v \rangle \]
so $I^* M_V I$ is compact (since the compact operators form a $*$-ideal) and self-adjoint, since $(ABC)^* = C^* B^* A^*$. It follows that $L^2(0,1)$ has an orthonormal basis of eigenfunctions $e_i$ for $I^* M_V I$ with eigenvalues
\[ \lambda_i = \langle I^* M_V I e_i, e_i \rangle = \langle M_V I e_i, I e_i \rangle \geq 0 \]
by the positivity of $V$. So
\[ B e_i = (1 + \lambda_i)^{\frac{1}{2}} e_i \]
defines, by continuous extension, a bounded operator on $L^2(0,1)$ such that
\[ B^2 = \text{Id} + I^* M_V I. \]

Or, without the compactness of $I$ (which can also be proved by checking tails in the Fourier basis) one needs to show that the spectrum of $A = \text{Id} + I^* M_V I$ is contained in $[0, \|A\|]$. This is not quite obvious, but follows from the positivity. Namely the operator $A - \frac{1}{2} \|A\| \text{Id}$ satisfies
\[ \frac{1}{2} \|A\| \|u\| \geq \langle (A - \frac{1}{2} \|A\| \text{Id}) u, u \rangle \geq -\frac{1}{2} \|A\| \]
so its spectrum is contained in $[-\frac{1}{2} \|A\|, \frac{1}{2} \|A\|]$. Then $B$ is well-defined by the functional calculus.