18.330 Numerical Analysis

Homework Assignment 6. (CORRECTED and due in class on THURSDAY 5/15/2003)

You may work in teams of up to three people and turn in one solution per team.

Problems courtesy of Prof. Alar Toomre.

Solve both problems.

1. (20 points) Determine the sag \( u(\pi/2, \pi/2) \) at the center of a uniformly loaded square membrane obeying the Poisson PDE

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 1
\]

with \( u = 0 \) along the four edges of length \( \pi \).

For that purpose build yourself a uniform \( N \times N \) mesh demanding

\[
u_{i,j-1} + u_{i,j+1} + u_{i-1,j} + u_{i+1,j} - 4u_{i,j} = h^2
\]

in the interior and \( u_{i0} = u_{iN} = u_{0j} = u_{N0} = 0 \) along the edges.

From this compute \( u_{N/2,N/2} \) to 9 decimals with the help of Gauss-Seidel-type successive-over-relaxations for \( N = 10, 14, 20, 28, 40, \) etc. using SOR coefficient \( \omega \approx 1.5 \). Follow that by ample Richardson extrapolations, since we are seeking this PDE answer really in the limit as \( N \) approaches infinity.

P.S.: Sag \( u(\pi/2, \pi/2) = -\pi^2/16 \) at the center of a circular membrane of diameter \( \pi \).

2. (15 points) Find the LU decomposition of the Pascal Matrix

\[
P = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 2 & 3 & 4 & 5 \\
1 & 3 & 6 & 10 & 15 \\
1 & 4 & 10 & 20 & 35 \\
1 & 5 & 15 & 35 & 70 \\
\end{bmatrix}
\]

and use it to ascertain the smallest eigenvalue of this matrix via repeated application – clearly explained and exemplified – of the so called inverse power method. Also track down the largest eigenvalue of \( P \) via the normal power method ... and thus rediscover a remarkable relationship between those two values.