MODEL ANSWERS TO HWK #3

1. (a)  
\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
4 & 5 & 2 & 1 & 3 & 6
\end{pmatrix}
\]

The cycle decomposition is
\[(1, 4)(2, 5, 3)\]
and so the order is 6.

(b)  
\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 \\
3 & 1 & 2 & 4 & 5
\end{pmatrix}
\]

The cycle decomposition is
\[(1, 3, 2)\]
and so the order is 3.

(c)  
\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 \\
1 & 4 & 3 & 2 & 5
\end{pmatrix}
\]

The cycle decomposition is
\[(2, 4)\]
and so the order is 2.

2. As $\sigma$ and $\tau$ are cycles, we may find integers $a_1, a_2, \ldots, a_k$ and $b_1, b_2, \ldots, b_l$ such that $\sigma = (a_1, a_2, \ldots, a_k)$ and $\tau = (b_1, b_2, \ldots, b_l)$. To say that $\sigma$ and $\tau$ are disjoint cycles is equivalent to saying that the two sets $S = \{a_1, a_2, \ldots, a_k\}$ and $T = \{b_1, b_2, \ldots, b_l\}$ are disjoint.

We want to prove that

$$\sigma \tau = \tau \sigma.$$

As both sides of this equation are permutations of the first $n$ natural numbers, it suffices to show that they have the same effect on any integer $1 \leq j \leq n$.

If $j$ is not in $S \cup T$, then there is nothing to prove; both sides clearly fix $j$. Otherwise $j \in S \cup T$. By symmetry we may assume $j \in S$. As $S$ and $T$ are disjoint, it follows that $j \notin T$.

As $j \in S$, $j = a_i$, some $i$. Then $\sigma(a_i) = a_{i+1}$, where we take $i + 1$ modulo $k$ (that is, we adopt the convention that $a_{k+1} = a_1$). In this case $a_{i+1} \in S$ so $a_{i+1} \notin T$ as well. Thus both sides send $j = a_i$ to $a_{i+1}$. 


Thus both sides have the same effect on $j$, regardless of $j$ and so
\[ \sigma \tau = \tau \sigma. \]

3. (i) \[ (1, 3, 4, 2)(5, 7, 9), \]
which has order 12.
(ii) \[ (1, 7)(2, 6)(3, 5), \]
which has order 2.
(iii) \[ (1, 6)(2, 5)(3, 7), \]
which has order 2.

4. \[ (2, 4, 1)(3, 5, 7, 6) = (2, 1)(2, 4)(3, 6)(3, 7)(3, 5). \]
The order is 12.
(f) \[ (1, 4, 2, 5, 3) = (1, 3)(1, 5)(1, 2)(1, 4). \]
The order is 5.

5. The conjugate is \((2, 7, 5, 3)(1, 6, 4)\). The order of $\sigma$ is 12 and the order of $\tau$ is three.

6. There are quite a few possibilities for $\tau$. One obvious one is
\[ \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 1 & 2 & 5 & 4 & 7 & 6 \end{pmatrix}. \]

7. Let $H = \langle (1, 2)(1, 2, 3, \ldots, n) \rangle$. We want to show that $H$ is the whole of $S_n$. As the transpositions generate $S_n$, it suffices to prove that every transposition is in $H$.
Now the idea is that it is very hard to compute products in $S_n$, but it is easy to compute conjugates. So instead of using the fact that $H$ is closed under products and inverses, let us use the fact that it is closed under taking conjugates (clear, as a conjugate is a product of elements of $H$ and their inverses). Since conjugation preserves cycle type, we start with the transposition $\sigma = (1, 2)$ (in fact this is the only place to start).
To warm up, consider conjugating $\sigma$ with $\tau = (1, 2, 3, \ldots, n)$. The conjugate is $(2, 3)$. Thus $H$ must contain $(2, 3)$.
Given that $H$ contains $(2, 3)$ it must contain the conjugate of $(2, 3)$ by $\tau$, which is $(3, 4)$ (or what comes to the same thing, $H$ must contain the conjugate of $(1, 2)$ by $\tau^2$).
Continuing in this way, it is clear that $H$ (by an easy induction in fact) must contain every transposition of the form $(i, i + 1)$ and of course the last one, $(n, 1) = (1, n)$.

There are now two ways to show that $H = S_n$.

The first is to complete the proof that $H$ must contain every transposition. First, let us try to show that $H$ contains every transposition of the form $(1, i)$. For example, to get $(1, 3)$, start with $(1, 2)$ and conjugate it by $(2, 3)$. Suppose, by way of induction, that $H$ contains $(1, i)$. Then $H$ must contain the conjugate of $(1, i)$ by $(i, i + 1)$ which is $(1, i + 1)$. Thus by induction $H$ contains every transposition of the form $(1, i)$.

Now we are almost home. First note that $H$ must contain every transposition of the form $(2, j)$. Indeed $(2, j)$ is the conjugate of $(1, j)$ by the transposition $(1, 2)$.

Now consider an arbitrary transposition $(i, j)$. This is the conjugate of $(1, 2)$ by the element $(1, i)(2, j)$. Thus $H$ contains every transposition and so $H = S_n$. This completes the first proof.

For the second proof we consider the second proof that the transpositions generate $S_n$. We need to show that we can put any deck of cards, in arbitrary order, into the standard order by only switching adjacent cards. By induction on $i$, we may assume that the first $i$ cards are in the correct order. We may suppose that the $i$th card is in position $j > i$ (if it is in position $i$ there is nothing to do). We need to check that we can put card $i$ in the $j$th position into the $i$th position, without moving the first $i - 1$ cards. If we switch cards $j$ and $j - 1$, then the $i$th card is now in the $j - 1$th position. Therefore switching cards cards $j$ and $j - 1$, then cards $j - 1$ and $j - 2$, . . . , and then cards $i + 1$ and $i$, card $i$ now occupies the $i$th position and we haven’t moved the first $i - 1$ cards (note though that the card that was in the $i$th position is not now in the $j$th position; it is in position $i + 1$).

This completes the induction on $i$. Thus one can undo any permutation just using the transpositions $(i, i + 1)$ which is to say that these transpositions generate $S_n.$