THIRD PRACTICE MIDTERM B  
MATH 18.02, MIT, AUTUMN 12

You have 50 minutes. This test is closed book, closed notes, no calculators.

There are 6 problems, and the total number of points is 100. Show all your work. Please make your work as clear and easy to follow as possible.

Name: ____________________________

Signature: ________________________

Student ID #: _____________________

Recitation instructor: ______________

Recitation Number+Time: _____________

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1. (15pts) Let \((\bar{x}, \bar{y})\) be the centre of mass of the triangle with vertices at \((-2, 0), (0, 1)\) and \((2, 0)\) (assume uniform density \(\delta = 1\)).

(i) Express \(\bar{y}\) in terms of an integral.

\[\bar{y} = \frac{1}{A} \int\int_R y \, dA = \frac{1}{2} \int_0^1 \int_{-2+2y}^{2-2y} y \, dx \, dy.\]

(ii) Find \(\bar{x}\).

\[\text{Solution: } \bar{x} = 0 \text{ by symmetry.}\]
2. (15pts) Find the moment of inertia about the origin of the half disk 
\(x^2 + y^2 < a^2, \ x > 0\), where the density \(\delta = x^2\).

Solution: Let \(R\) be the given region. We have

\[
\iint_R x^2(x^2 + y^2) \, dA = \int_{-\pi/2}^{\pi/2} \int_0^a r^5 \cos^2 \theta \, dr \, d\theta.
\]

The inner integral is

\[
\int_0^a r^5 \cos^2 \theta \, dr = \left[ \frac{r^6}{6} \cos^2 \theta \right]_0^a = \frac{a^6}{6} \cos^2 \theta.
\]

The outer integral is

\[
\int_{-\pi/2}^{\pi/2} \frac{a^6}{6} \cos^2 \theta \, d\theta = \frac{a^6}{12} \left[ \theta + \frac{1}{2} \sin 2\theta \right]_{-\pi/2}^{\pi/2} = \frac{\pi a^6}{12}.
\]
3. (10pts) For which value of $a$ is the vector field

$$\vec{F} = \left(axy - \frac{1}{x}\right)i + \left(x^2 - \frac{1}{y}\right)j$$

a gradient vector field?

Solution: We have

$$M = axy - \frac{1}{x} \quad \text{and} \quad N = x^2 - \frac{1}{y}.$$  

$\vec{F}$ is a gradient vector field if and only if

$$ax = M_y = N_x = 2x.$$ 

So $\vec{F}$ is a gradient vector field if and only if $a = 2$.  

4. (20pts) Let \( \vec{F} = 2y \hat{i} - x \hat{j} \). Let \( C \) be the curve \( y = x^2 \) starting at \( (0, 0) \) and ending at \( (1, 1) \).
(a) Compute the work done on a particle that moves along \( C \).

Solution: Let \( x(t) = t \) and \( y(t) = t^2 \), \( 0 \leq t \leq 1 \). Then
\[
\vec{F} = (2t^2, -t) \quad \text{and} \quad d\vec{r} = (1, 2t) \, dt.
\]
The work done is
\[
\int_C \vec{F} \cdot d\vec{r} = \int_0^1 (2t^2, -t) \cdot (1, 2t) \, dt = 0.
\]

(b) Compute the flux of \( \vec{F} \) across \( C \).

Solution: Let \( x(t) = t \) and \( y(t) = t^2 \), \( 0 \leq t \leq 1 \). Then
\[
\vec{F} = (2t^2, -t) \quad dx = dt \quad \text{and} \quad dy = 2t \, dt.
\]
The flux is
\[
\int_C \vec{F} \cdot \hat{n} \, ds = \int_C x \, dx + 2y \, dy = \int_0^1 (t + 4t^3) \, dt = \left[ \frac{t^2}{2} + t^4 \right]_0^1 = \frac{3}{2}.
\]
5. (20pts) (i) Express the work done by the force field

\[ \mathbf{F} = (5x + 3y)\mathbf{i} + (1 + \cos y)\mathbf{j} \]

on a particle going counterclockwise once around the unit circle centred at the origin in the form

\[ \int_{a}^{b} f(t) \, dt. \]

Solution: Parametrise the circle by \( x(t) = \cos t, y(t) = \sin t, 0 \leq t \leq 2\pi \). In this case

\[ \mathbf{F} = (5\cos t + 3\sin t, 1 + \cos(\sin t)) \]

and \( d\mathbf{r} = (-\sin t, \cos t) \, dt \).

Hence

\[ \oint_{C} \mathbf{F} \cdot d\mathbf{r} = \int_{0}^{2\pi} -5\cos t \sin t - 3\sin^2 t + \cos t + \cos t \cos(\sin t) \, dt. \]

(ii) Evaluate the work done by using Green’s theorem.

Solution:

\[ \oint_{C} \mathbf{F} \cdot d\mathbf{r} = \iint_{R} -3 \, dA = \iint_{R} -3 \, dA = -3\pi. \]
6. (20pts) Consider the region \( R \) enclosed by the \( x \)-axis, \( x = 1 \) and \( y = x^3 \). Let \( \vec{F} = (1 + y^2) \hat{j} \).

(i) Use the normal form of Green’s theorem to find the flux of \( \vec{F} \) out of \( R \).

**Solution:**

\[
\oint_C \vec{F} \cdot \hat{n} \, ds = \iint_R \text{div} F \, dA = \int_0^1 \int_0^{x^3} 2y \, dy \, dx.
\]

The inner integral is

\[
\int_0^{x^3} 2y \, dy = \left[ y^2 \right]_0^{x^3} = x^6.
\]

So the outer integral is

\[
\int_0^1 x^6 \, dx = \left[ \frac{x^7}{7} \right]_0^1 = \frac{1}{7}.
\]

(ii) Find the flux across the horizontal side \( C_1 \) of \( R \) and the vertical side \( C_2 \) of \( R \).

**Solution:** Parametrise \( C_1 \) by \( x(t) = t, \ y(t) = 0, \ 0 \leq t \leq 1 \). Then

\[
\vec{F} = \hat{j} \quad \text{and} \quad \hat{n} = -\hat{j}.
\]

So

\[
\int_{C_1} \vec{F} \cdot \hat{n} \, ds = \int_0^1 -1 \, dt = -1.
\]

Along \( C_2 \), \( \vec{F} \) is parallel to \( \hat{j} \) and \( \hat{n} \) is parallel to \( \hat{i} \), so the flux across \( C_2 \) is zero.

(iii) Find the flux across the third side \( C_3 \).

**Solution:**

\[
\frac{1}{7} = \oint_C \vec{F} \cdot \hat{n} \, ds = \int_{C_1} \vec{F} \cdot \hat{n} \, ds + \int_{C_2} \vec{F} \cdot \hat{n} \, ds + \int_{C_3} \vec{F} \cdot \hat{n} \, ds.
\]

So the flux across \( C_3 \) is

\[
\frac{1}{7} + 1 = \frac{8}{7}.
\]