FIRST PRACTICE MIDTERM B
MATH 18.02, MIT, AUTUMN 12

You have 50 minutes. This test is closed book, closed notes, no calculators.

There are 5 problems, and the total number of points is 100. Show all your work. Please make your work as clear and easy to follow as possible.

Name:__________________________
Signature:_______________________
Student ID #:____________________
Recitation instructor:_____________
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1. (20pts) The unit cube lies in the first octant in \( \mathbb{R}^3 \), so that one vertex is at the origin. Let \( Q \) be the vertex diagonally opposite the origin and let \( R \) be the midpoint of a face not containing the origin.

(i) Express \( \vec{Q} \) and \( \vec{R} \) in terms of \( \hat{i}, \hat{j} \) and \( \hat{k} \) (there are three choices for \( R \); pick one).

**Solution:**

\[
\vec{Q} = \hat{i} + \hat{j} + \hat{k} \quad \text{and} \quad \vec{R} = \hat{i} + \frac{1}{2}\hat{j} + \frac{1}{2}\hat{k}.
\]

(ii) Find the cosine of the angle between \( \vec{Q} \) and \( \vec{R} \).

**Solution:**

\[
\cos \theta = \frac{\langle 1, 1, 1 \rangle \cdot \langle 1, 1/2, 1/2 \rangle}{|\langle 1, 1, 1 \rangle||\langle 1, 1/2, 1/2 \rangle|} = \frac{2}{\sqrt{3} \sqrt{3/2}} = \frac{2\sqrt{2}}{3}.
\]
2. (20pts) (i) Let
\[ A = \begin{pmatrix} 1 & 1 & -3 \\ 1 & 2 & 0 \\ -1 & 0 & 4 \end{pmatrix} \]
then \( \det(A) = -2 \) and
\[ A^{-1} = \begin{pmatrix} -4 & a & b \\ 2 & -1/2 & 3/2 \\ -1 & 1/2 & -1/2 \end{pmatrix}. \]
Find \( a \) and \( b \).

Solution:
We know that \( AA^{-1} = I_3 \). Comparing entries in the first row second column we get
\[ a - 1/2 - 3/2 = 0 \quad \text{so that} \quad a = 2. \]
Comparing entries in the first row third column we get
\[ b + 3/2 + 3/2 = 0 \quad \text{so that} \quad b = -3. \]

(ii) Solve the system \( A\vec{x} = \vec{b} \), where
\[ \vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{and} \quad \vec{b} = \begin{pmatrix} 4 \\ 3 \\ -3 \end{pmatrix}. \]

Solution:
\[ \vec{x} = A^{-1}\vec{b} = \begin{pmatrix} -4 & 2 & -3 \\ 2 & -1/2 & 3/2 \\ -1 & 1/2 & -1/2 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \\ -3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \]

(iii) In the matrix \( A \), replace the entry \(-3\) in the upper-right corner by \( c \). Find a value of \( c \) for which the resulting matrix \( M \) is not invertible.
For this value of \( c \) the system \( M\vec{x} = \vec{0} \) has other solutions than the obvious one \( \vec{x} = \vec{0} \); find such a solution by using vector operations.

Solution: \( M \) invertible if and only if \( \det M \neq 0 \);
\[ 0 = \begin{vmatrix} 1 & 1 & c \\ 1 & 2 & 0 \\ -1 & 0 & 4 \end{vmatrix} = 2\begin{vmatrix} 0 & 1 \\ 4 & -1 \end{vmatrix} + c\begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} = 8 - 4 + 2c. \]
\[ c = -2. \]
Cross product is a solution of homogeneous,
\[ \begin{vmatrix} i & j & k \\ 1 & 2 & 0 \\ -1 & 0 & 4 \end{vmatrix} = i\begin{vmatrix} 0 & 1 \\ 4 & -1 \end{vmatrix} - j\begin{vmatrix} 1 & 0 \\ -1 & 0 \end{vmatrix} + k\begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} = 8i - 4j + 2k. \]
3. (20pts) Find the equation of the plane containing the point \( P_0 = (-1, 1, 1) \) and the line given as the intersection of the two planes
\[
\begin{align*}
2x - y + z &= -1 \\
x + y + z &= 3.
\end{align*}
\]

Solution:
Find two points on the line. Intersect the line with the plane \( x = 0 \) and \( x = -2 \). If \( x = 0 \), we have
\[
\begin{align*}
-y + z &= -1 \\
y + z &= 3.
\end{align*}
\]
Adding we get \( 2z = 2 \), so that \( z = 1 \). But then \( y = 2 \). \( P_1 = (0, 2, 1) \) is a point on the line. If \( x = -2 \) we have
\[
\begin{align*}
-y + z &= 3 \\
y + z &= 5.
\end{align*}
\]
Adding we get \( 2z = 8 \), so that \( z = 4 \). But then \( y = 1 \). Two points on the plane are \( P_1 = (0, 2, 1) \) and \( P_2 = (-2, 1, 4) \).

\[
\vec{v} = \overrightarrow{P_0P_1} = \langle 1, 1, 0 \rangle \quad \text{and} \quad \vec{w} = \overrightarrow{P_0P_2} = \langle -1, 0, 3 \rangle
\]
are two vectors parallel to the plane. The cross product \( \vec{v} \times \vec{w} \) is a normal vector to the plane,
\[
\begin{vmatrix}
i & j & k \\ 1 & 1 & 0 \\ -1 & 0 & 3
\end{vmatrix} = \hat{i} \begin{vmatrix} 1 & 0 \\ 0 & 3
\end{vmatrix} - \hat{j} \begin{vmatrix} 1 & 0 \\ -1 & 3
\end{vmatrix} + \hat{k} \begin{vmatrix} 1 & 1 \\ -1 & 0
\end{vmatrix} = 3\hat{i} - 3\hat{j} + \hat{k}.
\]
Hence \( \vec{n} = \langle 3, -3, 1 \rangle \) is a vector normal to the plane.
\[
\langle x+1, y-1, z-1 \rangle \cdot \langle 3, -3, 1 \rangle = 0 \quad \text{so that} \quad 3(x+1) - 3(y-1) + (z-1) = 0.
\]
Rearranging, we get
\[
3x - 3y + z = -5.
\]
4. (20pts)

(i) Find the area of the triangle with vertices $P_0 = (1, -1, 2)$, $P_1 = (2, 1, -3)$ and $P_2 = (3, 1, -1)$.

**Solution:**

Let $\vec{v} = \overrightarrow{P_0P_1} = \langle 1, 2, -5 \rangle$ and $\vec{w} = \langle 2, 2, -3 \rangle$. Then

$$\vec{v} \times \vec{w} = \begin{vmatrix}
i & j & k \\
1 & 2 & -5 \\
2 & 2 & -3 \\
\end{vmatrix} = i \begin{vmatrix}2 & -5 \\
2 & -3 \\
\end{vmatrix} - j \begin{vmatrix}1 & -5 \\
2 & 2 \\
\end{vmatrix} + k \begin{vmatrix}1 & 2 \\
2 & 2 \\
\end{vmatrix} = 4i - 7j - 2k.$$

The area of the triangle is half the magnitude of the cross product

$$\frac{1}{2} (4^2 + 7^2 + 2^2)^{1/2} = \frac{1}{2} \sqrt{69}.$$  

(ii) Find the equation of the plane containing these points.

**Solution:**

Let $P = \langle x, y, z \rangle$. Then $\overrightarrow{P_0P} = \langle x - 1, y + 1, z - 2 \rangle$ is orthogonal to $\vec{n} = \vec{v} \times \vec{w} = 4i - 7j - 2k$. Therefore

$$0 = \overrightarrow{P_0P} \cdot \vec{n} = \langle x - 1, y + 1, z - 2 \rangle \cdot \langle 4, -7, -2 \rangle = 4(x - 1) - 7(y + 1) - 2(z - 2).$$

Rearranging, we get

$$4x - 7y - 2z = 7.$$  

(iii) What is the shortest distance between the plane and the point $(1, 2, 3)$?

**Solution:**

The line through $(1, 2, 3)$ and parallel to $\vec{n} = \langle 4, -7, -2 \rangle$ intersects the plane at the closest point $Q$. This line is

$$\vec{r}(t) = \langle 1, 2, 3 \rangle + t \langle 4, -7, -2 \rangle = \langle 1 + 4t, 2 - 7t, 3 - 2t \rangle.$$  

It lies on the plane when

$$4(1 + 4t) - 7(2 - 7t) - 2(3 - 2t) = 7 \quad \text{so that} \quad 69t - 16 = 7.$$  

Thus $t = \frac{1}{3}$ and the point $Q = \langle 7/3, -1/3, 7/3 \rangle$. The distance is then

$$|\overrightarrow{PQ}| = |\langle 4/3, -7/3, -2/3 \rangle| = \frac{1}{3} (4^2 + 7^2 + 2^2)^{1/2} = \frac{1}{3} \sqrt{69}.$$  

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5. (20pts)
(i) Let \( \vec{r}(t) \) be the position vector of a particle in \( \mathbb{R}^3 \). Give a formula for
\[
\frac{d(\vec{r} \cdot \vec{r})}{dt}
\]
in vector coordinates.

Solution:
\[
\frac{d(\vec{r} \cdot \vec{r})}{dt} = 2\vec{r} \cdot \frac{d\vec{r}}{dt} = 2\vec{r} \cdot \vec{v}.
\]

(ii) Show that if \( \vec{r} \) has constant length, then \( \vec{r} \) and the velocity vector \( \vec{v} \) are orthogonal.

Solution:
If \( \vec{r} \) has constant length, then \( \vec{r} \cdot \vec{r} \) is constant and the first derivative is zero. But then \( \vec{r} \cdot \vec{v} = 0 \) and so \( \vec{r} \) and \( \vec{v} \) are orthogonal.

(iii) Let \( \vec{a} \) be the acceleration: still assuming that \( \vec{r} \) has constant length, and using vector differentiation, express the quantity \( \vec{r} \cdot \vec{a} \) in terms of the velocity vector only.

Solution:
Let us differentiate both sides of the equation
\[
\vec{r} \cdot \vec{v} = 0.
\]
We have
\[
0 = \frac{d(\vec{r} \cdot \vec{v})}{dt} = \vec{r} \cdot \frac{d\vec{v}}{dt} + \frac{d\vec{r}}{dt} \cdot \vec{v} = \vec{v} \cdot \vec{v} + \vec{r} \cdot \vec{a}.
\]
It follows that
\[
\vec{r} \cdot \vec{a} = -\vec{v} \cdot \vec{v}.
\]