1. (iv) If we write $\vec{F}(x, y) = M\hat{i} + N\hat{j}$, the curl of $\vec{F}$ is \( \text{curl} \vec{F} = N_x - M_y \). Since
$$N_x = \frac{(x^2 + y^2) - 2x^2}{(x^2 + y^2)^2} \quad \text{and} \quad M_y = -\frac{(x^2 + y^2) - 2y^2}{(x^2 + y^2)^2},$$
the curl of $\vec{F}$ vanishes everywhere $\vec{F}$ is defined.

(v) Let $C_1$ and $C_2$ be the curves in part (iii). Since
$$\int_{C_1} \vec{F} \cdot d\vec{r} \neq \int_{C_2} \vec{F} \cdot d\vec{r},$$
the vector field $\vec{F}$ is not conservative over its entire domain. However, it is conservative over the right half plane $x > 0$ since $\theta_2 - \theta_1$ only depends on the endpoints and if we have a loop, we obviously get zero.

2. (i) Note that the gradient of $r = \sqrt{x^2 + y^2}$ is
$$\nabla r = \frac{x}{\sqrt{x^2 + y^2}} \hat{i} + \frac{y}{\sqrt{x^2 + y^2}} \hat{j} = \frac{x}{r} \hat{i} + \frac{y}{r} \hat{j}.$$  

The curl of $\vec{F}$ is then
$$\text{curl} \vec{F} = \frac{\partial}{\partial x}(r^n y) - \frac{\partial}{\partial y}(r^n x) = nr^{n-1} \frac{x}{r} y - nr^{n-1} \frac{y}{r} x = 0.$$  

(ii) Let $g(r) = \frac{1}{n+2} r^{n+2}$. Then
$$\nabla g(r) = r^n (x\hat{i} + y\hat{j}),$$
so long as $n \neq -2$. If $n = -2$, consider instead $g(x, y) = \frac{1}{2} \ln(x^2 + y^2)$; here
$$\nabla g(r) = \frac{1}{x^2 + y^2} (x\hat{i} + y\hat{j}) = \frac{1}{r} (x\hat{i} + y\hat{j}).$$

3. (i) Consider the vector fields $\vec{F}_1 = \langle -y, 0 \rangle$ and $\vec{F}_2 = \langle 0, x \rangle$. Both of these vector fields satisfy curl $\vec{F} = 1$, and so applying Green’s theorem to $\vec{F}_1$ gives
$$\text{area}(R) = \int_R dA = \int_R \text{curl} \vec{F}_1 \ dA = - \oint_C y \ dx,$$
and similarly to $\vec{F}_2$ gives
$$\text{area}(R) = \int_R dA = \int_R \text{curl} \vec{F}_2 \ dA = \oint_C x \ dy.$$
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(ii) To obtain one arch we need the smallest positive $t$ with $y(t) = 0$. This gives $t = 2\pi$. Let

$$ \vec{r}_1(t) = \langle a(t - \sin t), a(1 - \cos t) \rangle \quad \text{and} \quad \vec{r}_2(t) = (2\pi - t, 0).$$

Then the curve $C = C_1 \cup C_2$ encloses $R$ with the opposite orientation, and so applying part (i) gives

$$ \text{area}(R) = -\int_C x \, dy = -\int_{C_1} x \, dy - \int_{C_2} x \, dy = a^2 \int_0^{2\pi} \sin^2 t - t \sin t \, dt = a^2 \left[ \int_0^{2\pi} (t \cos t) - \int_0^{2\pi} \cos t \, dt + \frac{1}{2} \int_0^{2\pi} (1 - \cos(2t)) \, dt \right] = 3\pi a^2.$$

4. (i) Observe that if

$$ \vec{F} = \langle x^2y + y^3 - y, 3x + 2y^2x + e^y \rangle \quad \text{then} \quad \text{curl} \vec{F} = 4 - (x^2 + y^2).$$

Therefore by Green’s theorem we have that if $C$ bounds the region $R$ then

$$ \oint_C \vec{F} \cdot dr = \iint_R \text{curl} \vec{F} \, dA.$$

So we want $R$ to be the region where the curl is at least zero, that is, we want $x^2 + y^2 \leq 4$. The boundary $C$ of this region is the circle of radius 2, centred at the origin.

(ii) Again by applying Green’s theorem we get that

$$ \int_C \vec{F} \cdot dr = \iint_R \text{curl} \vec{F} \, dA$$

$$ = \iint_{x^2 + y^2 \leq 4} (4 - x^2 - y^2) \, dA$$

$$ = \int_0^{2\pi} \int_0^2 (4 - r^2)r \, dr \, d\theta$$

$$ = 2\pi \left[ 2r^2 - \frac{1}{4} r^4 \right]_0^2$$

$$ = 8\pi.$$

5. (i) True. If $\vec{F} = \nabla f$ and $\vec{G} = \nabla g$ then $\vec{F} + \vec{G} = \nabla (f + g)$. 
(ii) True. If $\vec{F}$ is a gradient vector field then $\text{curl} \vec{F} = N_x - M_y = 0$. In particular $M_y(1, -1) = N_x(1, -1)$.

6. (i) Note that the normal vector to the unit circle is simply the radial vector $(x, y)$. We compute the flux

$$\vec{F} \cdot \vec{n} = \langle xy, y^2 \rangle \cdot (x, y) = y(x^2 + y^2).$$

We therefore see that $y \geq 0$, the upper half of the circle, contributes positively to the flux while $y \leq 0$, the lower half of the circle, contributes negatively to the flux.

(ii) Using the unit speed parametrization $\vec{r}(s) = (\cos s, \sin s)$ with $s \in [0, 2\pi]$ we can use part (i) to compute

$$\int_0^{2\pi} \vec{F} \cdot \vec{n} \, ds = \int_0^{2\pi} \sin s (\cos^2 s + \sin^2 s) \, ds = \int_0^{2\pi} \sin s \, ds = 0.$$

This gels with (i) because for each point $(x, y)$ on the unit circle the flux at the corresponding point $(x, -y)$ has equal magnitude but opposite sign. Hence, we expect the total flux to be zero.

(iii) Using Green’s theorem we get

$$\int_0^{2\pi} \vec{F} \cdot \vec{n} \, ds = \iint_{x^2+y^2 \leq 1} \text{div} \vec{F} \, dA$$

$$= \iint_{x^2+y^2 \leq 1} 3y \, dA = 0,$$

since $y$ is anti-symmetric about the $x$-axis and the unit circle is symmetric about the $x$-axis.