FOURTH MIDTERM
MATH 18.02, MIT, AUTUMN 12

You have 50 minutes. This test is closed book, closed notes, no calculators.

There are 5 problems, and the total number of points is 100. Show all your work. *Please make your work as clear and easy to follow as possible.*

Name:__________________________
Signature:_______________________
Student ID #:____________________
Recitation instructor:_____________
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1. (20pts) Let $R$ be the region in space which lies above the $xy$-plane and below the paraboloid $z = 1 - x^2 - y^2$. Calculate the moment of inertia about the $z$-axis; assume the density $\delta = 1$.

**Solution:**

$$I_z = \iiint_R x^2 + y^2 \, dV = \int_0^{2\pi} \int_0^1 \int_0^{1-r^2} r^3 \, dz \, dr \, d\theta.$$ 

The inner integral is

$$\int_0^{1-r^2} r^3 \, dz = \left[ r^3z \right]_0^{1-r^2} = r^3(1 - r^2).$$

The middle integral is

$$\int_0^1 r^3 - r^5 \, dr = \left[ \frac{r^4}{4} - \frac{r^6}{6} \right]_0^1 = \frac{1}{12}.$$

The outer integral is

$$\int_0^{2\pi} \frac{1}{12} \, d\theta = \frac{\pi}{6}.$$
2. (20pts) (i) A solid sphere of radius \( a \) is placed above the \( xy \)-plane so it is tangent at the origin and so the \( z \)-axis is a diameter. Give its equation in spherical coordinates.

\[ x^2 + (z - a)^2 = a^2 \quad \text{so that} \quad x^2 + z^2 = 2za. \]

As \( x^2 + z^2 = \rho^2 \) and \( z = \rho \cos \phi \), the equation of the sphere is

\[ \rho = 2a \cos \phi. \]

(ii) Give the equation of the horizontal plane \( z = a \) in spherical coordinates.

\[ \rho \cos \phi = a. \]

(iii) Set up a triple integral in spherical coordinates which gives the volume of the portion of the sphere \( S \) lying above the plane \( z = a \).

\[ \iiint_S 1 \, dV = \int_0^{2\pi} \int_0^{\pi/4} \int_{a \sec \phi}^{2a \cos \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta. \]
3. (20pts) Let $D$ be the disk described by $x^2 + y^2 \leq 4$ and $z = 2$. Let $T$ be the right circular cone formed by joining every point of $D$ to the origin, so that $(0,0,0)$ is the vertex of the cone. Assume that $T$ has constant density $\delta = 1$. Set up an iterated integral that gives the magnitude of the gravitational force acting on a unit mass at the origin.

Solution: If 

$$\vec{F} = \langle F_x, F_y, F_z \rangle,$$

is the force due to gravity then $F_x = F_y = 0$ by symmetry. We have

$$|\vec{F}| = F_z = \iiint_T \frac{Gz}{\rho^3} \, dV = G \int_0^{2\pi} \int_0^{\pi/4} \int_0^{2\sec \phi} \cos \phi \sin \phi \, d\rho \, d\phi \, d\theta.$$
4. (20pts) Find the flux of the vector field

\[ \vec{F} = y^4 \hat{i} - x^3 \hat{j} + z \hat{k}, \]

coming out of the unit sphere centred at the origin.

Solution: We apply the divergence theorem. Let \( S \) be the surface of the unit sphere, oriented outwards and let \( V \) the solid enclosed by \( S \). Then the flux coming out of the sphere is

\[
\oiint_{S} \vec{F} \cdot d\vec{S} = \iiint_{V} \text{div} \vec{F} \, dV = \iiint_{V} 1 \, dV = \frac{4\pi}{3}.
\]
5. (20pts) Let $\vec{F} = (y + z)\hat{i} - x\hat{j} + (7x + 5)\hat{k}$, be a vector field and let $S$ be the part of the surface $z = 9 - x^2 - y^2$ that lies above the $xy$-plane. Orient $S$ by using the outward normal vector. Find the outward flux of $\vec{F}$ across $S$.

Solution: Let $S'$ be the surface $x^2 + y^2 < 9$, $z = 0$, oriented upwards. By the divergence theorem

$$\oiint_{S-S'} \vec{F} \cdot d\vec{S} = \iiint_V \text{div} \vec{F} \, dV = \iiint_V 0 \, dV = 0.$$ 

So

$$\oiint_S \vec{F} \cdot d\vec{S} = \oiint_{S'} \vec{F} \cdot d\vec{S}.$$ 

The unit normal to $S'$ is $\hat{k}$. So

$$\vec{F} \cdot \hat{k} = 7x + 5.$$ 

We have

$$\oiint_{S'} \vec{F} \cdot d\vec{S} = \iint_{S'} 7x + 5 \, dA = \iint_{S'} 5 \, dA = 45\pi,$$

since $x$ is skew-symmetric about the $y$-axis.