FIRST MIDTERM
MATH 18.02, MIT, AUTUMN 12

You have 50 minutes. This test is closed book, closed notes, no calculators.

There are 5 problems, and the total number of points is 100. Show all your work. Please make your work as clear and easy to follow as possible.

Name:________________________

Signature:_______________________

Student ID #:_____________________

Recitation instructor:______________

Recitation Number+Time:____________

<table>
<thead>
<tr>
<th>Problem</th>
<th>Points</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>100</td>
<td></td>
</tr>
</tbody>
</table>
1. (20pts) Let $P = (1, 2, 3)$, $Q = (-1, 2, 1)$ and $R = (1, 1, -3)$.

(i) What is the cosine of the angle between $\overrightarrow{PQ}$ and $\overrightarrow{PR}$?

Solution:

$\overrightarrow{PQ} = \langle -2, 0, -2 \rangle$ and $\overrightarrow{PR} = \langle 0, -1, -6 \rangle$.

Hence

$$\cos \theta = \frac{\langle -2, 0, -2 \rangle \cdot \langle 0, -1, -6 \rangle}{|\langle -2, 0, -2 \rangle||\langle 0, -1, -6 \rangle|} = \frac{12}{\sqrt{8}\sqrt{37}} = \frac{6}{\sqrt{2}\sqrt{37}}.$$

(ii) If $\vec{r}(t) = \langle -3 \cos 2t, 3 \sin 2t, 8t \rangle$, then what is the speed at time $t$?

Solution:

$\vec{v}(t) = \frac{d\vec{r}(t)}{dt} = \langle 6 \sin 2t, 6 \cos 2t, 8 \rangle = 2\langle 3 \sin 2t, 3 \cos 2t, 4 \rangle$.

Speed is the magnitude of the velocity,

$$2(3^2 \sin^2 2t + 3^2 \cos^2 2t + 4^2)^{1/2} = 2(3^2 + 4^2)^{1/2} = 10.$$
2. (20 pts) (i) Let

\[ A = \begin{pmatrix} -3 & 3 & 3 \\ -3 & 4 & 3 \\ -3 & 3 & 4 \end{pmatrix} \]

then \( \det(A) = -3 \) and

\[ A^{-1} = \begin{pmatrix} -7/3 & 1 & 1 \\ -1 & a & b \\ -1 & 0 & 1 \end{pmatrix}. \]

Find \( a \) and \( b \).

**Solution:**

We have \( AA^{-1} = I_3 \). Comparing entries in the first row second column, we have

\[ -3 + 3a = 0 \quad \text{so that} \quad a = 1 \]

and comparing entries in the first row third column, we have

\[ -3 + 3b + 3 = 0 \quad \text{so that} \quad b = 0. \]

(ii) Solve the system \( A\vec{x} = \vec{b} \), where

\[ \vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{and} \quad \vec{b} = \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix}. \]

**Solution:**

\[ \vec{x} = A^{-1}\vec{b} = \begin{pmatrix} -7/3 & 1 & 1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}. \]

(iii) In the matrix \( A \), replace the entry \(-3\) in the lower-left corner by \( c \). Find a value of \( c \) for which the resulting matrix \( M \) is not invertible. For this value of \( c \) the system \( M\vec{x} = \vec{0} \) has other solutions than the obvious one \( \vec{x} = \vec{0} \); find such a solution by using vector operations.

**Solution:** \( M \) is invertible if and only if \( \det M \neq 0 \).

\[
\begin{vmatrix} -3 & 3 & 3 \\ -3 & 4 & 3 \\ c & 3 & 4 \end{vmatrix} = -3 \begin{vmatrix} 4 & 3 \\ 3 & 4 \end{vmatrix} -3 \begin{vmatrix} -3 & 3 \\ c & 4 \end{vmatrix} +3 \begin{vmatrix} -3 & 4 \\ c & 3 \end{vmatrix} = -21+9c+36-12c-27 = -12-3c.
\]

So \( c = -4 \). We are looking for a vector orthogonal to all three rows. Take the cross product of the first two rows,

\[
\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & 3 & 3 \\ -3 & 4 & 3 \end{vmatrix} = \hat{i} \begin{vmatrix} 3 & 3 \\ 4 & 3 \end{vmatrix} - \hat{j} \begin{vmatrix} -3 & 3 \\ -3 & 4 \end{vmatrix} + \hat{k} \begin{vmatrix} -3 & 3 \\ -3 & 4 \end{vmatrix} = -3\hat{i} - 3\hat{k}.
\]
3. (20pts) (i) (6 points) Find the area of the triangle whose vertices are

\[ P_0 = (1, 1, 1), \quad P_1 = (1, 2, 3) \text{ and } P_2 = (-1, -1, 2) \, . \]

**Solution:**

Let \( \vec{v} = \overrightarrow{P_0 P_1} = \langle 0, 1, 2 \rangle \) and \( \vec{w} = \overrightarrow{P_0 P_2} = \langle -2, -2, 1 \rangle \). Then

\[
\vec{v} \times \vec{w} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
0 & 1 & 2 \\
-2 & -2 & 1 \\
\end{vmatrix} = \hat{i} \begin{vmatrix} 1 & 2 \\
-2 & 1 \\
\end{vmatrix} - \hat{j} \begin{vmatrix} 0 & 2 \\
-2 & 1 \\
\end{vmatrix} + \hat{k} \begin{vmatrix} 0 & 1 \\
-2 & -2 \\
\end{vmatrix} = 5\hat{i} - 4\hat{j} + 2\hat{k}.
\]

The area of the triangle is half the magnitude of the cross product

\[
\frac{1}{2} (5^2 + 4^2 + 2^2)^{1/2} = \frac{1}{2} \sqrt{45} = \frac{3}{2} \sqrt{5}.
\]

(ii) (6 points) Find the equation of the plane containing these points.

**Solution:**

Let \( P = \langle x, y, z \rangle \). Then \( \overrightarrow{P_0 P} = \langle x - 1, y - 1, z - 1 \rangle \) is orthogonal to \( \vec{n} = \vec{v} \times \vec{w} = 5\hat{i} - 4\hat{j} + 2\hat{k} \). Therefore

\[
0 = \overrightarrow{P_0 P} \cdot \vec{n} = (x - 1, y - 1, z - 1) \cdot (5, -4, 2) = 5(x - 1) - 4(y - 1) + 2(z - 1).
\]

Rearranging, we get

\[ 5x - 4y + 2z = 3. \]

(iii) (8 points) Find the point of intersection of this plane and the line through the point \((-1, 2, -1)\), parallel to the vector \( \vec{v} = \langle 3, 2, 1 \rangle \).

**Solution:**

The parametric form of the line is

\[
\vec{r}(t) = \langle -1, 2, -1 \rangle + t \langle 3, 2, 1 \rangle = \langle -1 + 3t, 2 + 2t, -1 + t \rangle.
\]

Plug this into the equation for the plane and solve for \( t \),

\[
3 = 5(-1 + 3t) - 4(2 + 2t) + 2(-1 + t) = 9t - 15.
\]

Hence \( t = 2 \) and the point we are looking for is \( (5, 6, 1) \).
4. (20 pts) (i) (8 points) Show that the line \( \vec{r}(t) = \langle 1 - t, 2 + 3t, 2 + t \rangle \) and the plane \( 2x + y - z = 4 \) are parallel.

**Solution:**
The line is parallel to \( \vec{v} = \langle -1, 3, 1 \rangle \). A normal vector to the plane is \( \vec{n} = \langle 2, 1, -1 \rangle \). We have
\[
\vec{v} \cdot \vec{n} = \langle -1, 3, 1 \rangle \cdot \langle 2, 1, -1 \rangle = -2 + 3 - 1 = 0,
\]
so that the vector \( \vec{v} \) is indeed parallel to the plane.

(ii) (12 points) Find the distance between the line \( \vec{r}(t) = \langle 1 - t, 2 + 3t, 2 + t \rangle \) and the plane \( 2x + y - z = 4 \).

**Solution:**
Set \( t = 0 \) to get a point \( P = (1, 2, 2) \) of the line. If \( Q \) is the point on the plane which lies on the line through \( P \) and parallel to \( \vec{n} \), then \( |\vec{PQ}| \) is the distance between the plane and the line.

\[
\vec{Q}(t) = \langle 1, 2, 2 \rangle + t\langle 2, 1, -1 \rangle = \langle 1 + 2t, 2 + t, 2 - t \rangle,
\]
is the parametric form for the line through \( P \) parallel to \( \vec{n} \). This meets the plane when
\[
2(1 + 2t) + (2 + t) - (2 - t) = 4 \quad \text{so that} \quad 2 + 6t = 4,
\]
that is \( t = 1/3 \). Therefore \( Q = (5/3, 7/3, 5/3) \) and \( \vec{PQ} = \langle 2/3, 1/3, -1/3 \rangle \).

It follows that the distance is
\[
\frac{1}{3} |\langle 2, 1, -1 \rangle| = \frac{1}{3} (4 + 1 + 1)^{1/2} = \frac{\sqrt{6}}{3}.
\]
5. (20pts) (i) A wheel of radius $a$ is rolling along the ground, the $x$-axis, in the $xy$-plane. The wheel is rotating at 2 radians per second clockwise. At time $t = 0$, the bottom of the wheel is at the origin. Find the position vector $\mathbf{r}(t)$ of the point on the rim which is at the top of the wheel at time $t = 0$.

**Solution:**
Let $A$ be the point where the wheel touches the ground, $B$ be the centre of the wheel and $P$ be the position of the point on the rim. We have

\[
\mathbf{A} = (2at, 0), \quad \mathbf{AB} = (0, a) \quad \text{and} \quad \mathbf{BP} = (a \sin 2t, a \cos 2t).
\]

So

\[
\mathbf{r}(t) = \mathbf{A} + \mathbf{AB} + \mathbf{BP} = (2at + a \sin 2t, a + a \cos 2t).
\]

(ii) What is the speed of this point at time $t$?

**Solution:**

\[
\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt} = (2a + 2a \cos 2t, -2a \sin 2t).
\]

Speed is the magnitude of the velocity,

\[
2a \left( (1 + \cos 2t)^2 + \sin^2 2t \right)^{1/2} = 2a \left( 1 + 2 \cos 2t + \cos^2 2t + \sin^2 2t \right)^{1/2} = 2a(2 + 2 \cos 2t)^{1/2} = 2a(4 \cos^2 t)^{1/2} = 4a |\cos t|.
\]

(iii) How far does this point move in one revolution?

The distance travelled is the integral of the speed,

\[
s = 2 \int_0^{\pi/2} 4a \cos t \, dt = 8a.
\]