MODEL ANSWERS TO HWK #9

1. There are a number of ways to proceed; probably the most straightforward is to view the region $D$ as something of type 2:

$$
\int \int_D x + y \, dx \, dy = \int_{-1}^{2} \left( \int_{y^2-2y}^{2-y} x + y \, dx \right) \, dy
$$

$$
= \int_{-1}^{2} \left[ \frac{x^2}{2} + yx \right]_{y^2-2y}^{2-y} \, dy
$$

$$
= \int_{-1}^{2} \frac{(2 - y)^2}{2} + y(2 - y) - \frac{(y^2 - 2y)^2}{2} - y(y^2 - 2y) \, dy
$$

$$
= \int_{-1}^{2} -\frac{y^4}{2} + y^3 - \frac{y^2}{2} \, dy + 2 dy
$$

$$
= \left[ -\frac{y^5}{2 \cdot 5} + \frac{y^4}{4} - \frac{y^3}{2 \cdot 3} + 2y \right]_{-1}^{2}
$$

$$
= -\frac{2^4}{5} + 2^2 - \frac{2^2}{3} + 2^2 - \frac{1}{10} - \frac{1}{4} - \frac{1}{6} + 2
$$

$$
= \frac{99}{20}
$$

2. There are a number of ways to proceed; probably the most straightforward is to view the region $D$ as something of type 1:

$$
\int \int_D 3y \, dx \, dy = \int_{0}^{1} \left( \int_{x}^{3} 3y \, dy \right) \, dx + \int_{\frac{1}{2}}^{1} \left( \int_{x}^{x^{-1/2}} 3y \, dy \right) \, dx
$$

$$
= \int_{0}^{1} \left[ \frac{3y^2}{2} \right]_{x}^{3} \, dx + \int_{\frac{1}{2}}^{1} \left[ \frac{3y^2}{2} \right]_{x}^{x^{-1/2}} \, dx
$$

$$
= \int_{0}^{1} \frac{3^3}{2} - \frac{3x^2}{2} \, dx + \int_{\frac{1}{2}}^{1} \frac{3}{2x} - \frac{3x^2}{2} \, dx
$$

$$
= \left[ \frac{3^3x}{2} - \frac{x^3}{2} \right]_{0}^{3} + \left[ \frac{3}{2} \ln x - \frac{x^3}{2} \right]_{\frac{1}{2}}^{1}
$$

$$
= \frac{3}{2} - \frac{1}{2 \cdot 3^6} - \frac{1}{2} + 3 \ln 3 + \frac{1}{2 \cdot 3^6}
$$

$$
= 1 + 3 \ln 3
$$
The region in question is bounded by the curves $x = 0$, $y = 0$ and $y^2 = 4 - x$. So, reversing the order of integration, we get

$$
\int_0^4 \left( \int_0^{\sqrt{4-x}} x \, dy \right) \, dx = \int_0^4 x [y]^{4-x} \, dx
$$

$$
= \int_0^4 x \sqrt{4-x} \, dx
$$

$$
= \left[ -\frac{2x}{3} (4-x)^{3/2} \right]_0^4 + \int_0^4 \frac{2}{3} (4-x)^{3/2} \, dx
$$

$$
= \left[ -\frac{4}{3 \cdot 5} (4-x)^{5/2} \right]_0^4
$$

$$
= \frac{2^7}{3 \cdot 5}.
$$
4. 
\[
\int_8^0 \left( \int_0^{\sqrt[3]{y/3}} y \, dx \right) \, dy + \int_8^{12} \left( \int_0^{\sqrt[5]{y/3}} y \, dx \right) \, dy = \int_0^2 \left( \int_{3x^2}^{x^2+8} y \, dy \right) \, dx \\
= \int_0^2 \left[ \frac{y^2}{2} \right]_{3x^2}^{x^2+8} \, dx \\
= \int_0^2 \left( \frac{x^5}{2} + \frac{18x^3}{3} - \frac{9x^5}{2} \right) \, dx \\
= \left[ \frac{x^6}{3} + 2x^4 - \frac{9x^5}{2} \right]_0 = 896 \cdot \frac{1}{15}.
\]

5. This is a region of type 4; we view this as an elementary region of type 1. The projection of \( W \) onto the \( xy \)-plane is the elementary region of type 2 bounded by \( y = x^2 \) and \( y = 9 \).

\[
\iiint_W 8xyz \, dx \, dy \, dz = \int_3^3 \left( \int_x^9 \left( \int_0^{9-y} 8xyz \, dz \right) \, dy \right) \, dx \\
= 8 \int_3^3 \left( \int_x^9 \left( \int_0^{9-y} z \, dz \right) \, dy \right) \, dx \\
= 8 \int_3^3 \left( \int_x^9 \left[ z^2 \right]_0^{9-y} \, dy \right) \, dx \\
= 8 \int_3^3 \left( \int_x^9 \frac{y(9-y)^2}{2} \, dy \right) \, dx \\
= 4 \int_3^3 \left( \int_x^9 \frac{81y - 18y^2 + y^3}{2} \, dy \right) \, dx \\
= 4 \int_3^3 \left[ \frac{81y^2}{2} - 6y^3 + \frac{y^4}{4} \right]_x^9 \, dx \\
= 4 \left\{ \frac{3^8}{2} - 2 \cdot 3^7 + \frac{3^8}{4} \right\} x - \frac{81x^3}{2} + 6x^7 - \frac{x^9}{4} \, dx \\
= 0.
\]
as $x$, $x^3$, $x^7$ and $x^9$ are all odd functions. In retrospect, we could have
decide very early on that the integral is zero;

$$J(x) = \int_{x^2}^{x^9} y \left( \int_{0}^{9-y} z \, dz \right) \, dy,$$

is clearly an even function of $x$, so that $xJ(x)$ is an odd function.

6. This is a region of type 4; we view this as an elementary region of
type 1. The projection of $W$ onto the $xy$-plane is the elementary region
of type 2 bounded by $x = 0$, $y = 3$ and $y = x$.

$$\int\int\int_W z \, dx \, dy \, dz = \int_{0}^{3} \left( \int_{x}^{3} \left( \int_{0}^{\sqrt{9-y^2}} z \, dz \right) dy \right) \, dx$$

$$= \int_{0}^{3} \left( \int_{x}^{3} \left[ \frac{z^2}{2} \right]_{0}^{\sqrt{9-y^2}} dy \right) \, dx$$

$$= \int_{0}^{3} \left( \int_{x}^{3} \frac{9-y^2}{2} dy \right) \, dx$$

$$= \frac{1}{2} \int_{0}^{3} \left[ 9y - \frac{y^3}{3} \right]_{x}^{3} dx$$

$$= \frac{1}{2} \int_{0}^{3} 18 - 9x + \frac{x^3}{3} \, dx$$

$$= \frac{1}{2} \left[ 18x - \frac{9x^2}{2} + \frac{x^4}{12} \right]_{0}^{3}$$

$$= 3^3 - \frac{3^4}{4} + \frac{3^3}{8}$$

$$= \frac{3^3}{8}(8 - 6 + 1)$$

$$= \frac{81}{8}.$$
7. This is the region bounded by the planes $y = \pm 1$, $x = y^2$, $z = 0$ and $x + z = 1$. So the other five ways to write this region are:

\[
\begin{align*}
\int_0^1 \left( \int_{-\sqrt{\frac{1}{x}}}^{\sqrt{\frac{1}{x}}} \left( \int_0^{1-x} f(x, y, z) \, dz \right) \, dy \right) \, dx \\
\int_0^1 \left( \int_{-\sqrt{\frac{1}{x}}}^{\sqrt{\frac{1}{x}}} \left( \int_0^{1-x} f(x, y, z) \, dy \right) \, dz \right) \, dx \\
\int_0^1 \left( \int_{-\sqrt{\frac{1}{x}}}^{\sqrt{\frac{1}{x}}} \left( \int_0^{1-z} f(x, y, z) \, dx \right) \, dy \right) \, dz \\
\int_{-1}^1 \left( \int_0^{1-y^2} \left( \int_0^{1-z} f(x, y, z) \, dx \right) \, dz \right) \, dy \\
\int_0^1 \left( \int_{\sqrt{1-z}}^{1-z} \left( \int_{y^2}^{1-z} f(x, y, z) \, dx \right) \, dy \right) \, dz.
\end{align*}
\]

8. $T$ is a linear transformation; therefore it takes straight lines to straight lines. So $D$ is the parallelogram with vertices

$T(0, 0) = (0, 0) \quad T(1, 3) = (11, 2) \quad T(-1, 2) = (4, 3) \quad T(0, 5) = (15, 5)$.

9. Since $T$ is supposed to take $(0, 5)$ to $(4, 1)$, it must take $(0, 1)$ to $(4/5, 1/5)$. Since $T$ is supposed to take $(-1, 3)$ to $(3, 2)$ and $(1, 2)$ to $(1, -1)$ it should take

$$(5, 0) = 3(1, 2) - 2(-1, 3),$$

to

$$3(3, 2) - 2(1, -1) = (7, 8).$$

Therefore

$$T(1, 0) = (7/5, 8/5).$$

Therefore

$$T(u, v) = \begin{pmatrix} 7/5 & 4/5 \\ 8/5 & 1/5 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$ 

10. We have $x = u$ and $y = (v + u)/2$. The Jacobian is

$$\frac{\partial(x, y)}{\partial(u, v)}(u, v) = \begin{vmatrix} 1 & 0 \\ 1/2 & 1/2 \end{vmatrix} = \frac{1}{2}.$$
This is nowhere zero. As the map is linear, it follows that the map is injective, and so by the Inverse function theorem it defines a diffeomorphism. Therefore,

\[ \int_0^2 \left( \int_{x/2}^{(x/2)+1} x^5(2y - x)e^{(2y-x)^2} \, dx \right) \, dy = \frac{1}{2} \int_0^2 \left( \int_0^2 u^5 v e^{v^2} \, dv \right) \, du \]
\[ = \frac{1}{4} \int_0^2 u^5 \left[ e^{x^2} \right]_0^2 \, du \]
\[ = \frac{e^4 - 1}{4} \int_0^2 u^5 \, du \]
\[ = \frac{e^4 - 1}{24} [u^6]_0^1 \]
\[ = \frac{8(e^4 - 1)}{3}. \]

11. Let \( u = 2x + y \) and \( v = x - y \). Then

\[ \frac{\partial (u, x)}{\partial (x, y)}(x, y) = \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -3. \]

So

\[ \frac{\partial (x, y)}{\partial (u, v)}(u, v) = -\frac{1}{3}. \]

This is nowhere zero. As the map is linear, it follows that the map is injective, and so by the Inverse function theorem it defines a diffeomorphism. Therefore,

\[ \int_D (2x + y)^2 e^{x-y} \, dx \, dy = \frac{1}{3} \int_1^4 \left( \int_{-1}^1 u^2 e^v \, dv \right) \, du \]
\[ = \frac{1}{3} \int_1^4 u^2 [e^v]_{-1}^1 \, du \]
\[ = \frac{e - e^{-1}}{3} \int_1^4 u^2 \, du \]
\[ = \frac{e - e^{-1}}{9} [u^3]_1^4 \]
\[ = 7(e - e^{-1}). \]

12. Let \( u = y + 2x \) and \( v = 2y - x \). Then \( D^* \) is the region \([0, 5] \times [-5, 0]\), in \( uv\)-coordinates.

\[ \frac{\partial (u, x)}{\partial (x, y)}(x, y) = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 5. \]
So

\[ \frac{\partial (x, y)}{\partial (u, v)} (u, v) = \frac{1}{5}. \]

This is nowhere zero. As the map is linear, it follows that the map is injective, and so by the Inverse function theorem it defines a diffeomorphism. Therefore

\[
\int_0^5 \int_0^5 2x + y - 3 \ dx \ dy = \frac{1}{5} \int_0^5 \left( \int_{-5}^0 u - 3 \ dv + 6 \right) \ du
\]

\[
= \frac{1}{5} \int_0^5 (u - 3) \left[ \ln(v + 6) \right]_{-5}^0 \ du
\]

\[
= \frac{\ln 6}{5} \int_0^5 (u - 3) \ du
\]

\[
= \frac{\ln 6}{5} \left[ \frac{u^2}{2} - 3u \right]_0^5
\]

\[
= \ln 6 \left( \frac{5}{2} - 3 \right)
\]

\[
= -\frac{\ln 6}{2}.
\]