7. Vector spaces

We now abstract what we mean by a vector space. One key example to keep in mind is the set of polynomials of bounded degree. One can think of their coefficients as giving points in some $F^d$.

**Definition 7.1.** Let $F$ be a field. A **vector space** over $F$ consists of a set $V$ (the elements of which are called vectors) and two operations, addition and scalar multiplication,

$$+: V \times V \rightarrow V \quad \text{and} \quad \cdot: F \times V \rightarrow V,$$

which obey certain axioms. $(V, +)$ is an abelian (aka commutative) group under addition,

1. **Addition is associative.** That is for every $u, v$ and $w \in V$,
   $$u + (v + w) = u + (v + w).$$

2. **There is an identity element under addition.** This element is called the zero vector, it is denoted $0 \in V$ and for every element $v \in F$,
   $$0 + v = v + 0 = v.$$

3. **Every element has an additive inverse.** That is given $v \in V$ there is an element $-v \in V$ and
   $$v + (-v) = -v + v = 0.$$

4. **Addition is commutative.** That is given $v$ and $w \in V$,
   $$v + w = w + v.$$

There has to be some compatibility between the operation of multiplication of scalars (that is multiplication in the field) and scalar multiplication (that is multiplying a vector by a scalar):

5. **Given a vector and two scalars $\lambda$ and $\mu$,**
   $$\lambda(\mu v) = (\lambda \mu)v.$$

Finally we require that addition of vectors and scalar multiplication satisfy the distributive law:

6. **Given vectors $v$ and $w$ and a scalar $\lambda$,**
   $$\lambda(v + w) = \lambda v + \lambda w.$$

*Similarly, given two scalars $\lambda$ and $\mu$ and a vector $v$,*

$$\lambda + \mu)v = \lambda v + \mu v.$$
We have already seen a plenitude of examples. $F^d$ is a vector space for every $d$. Adopting a minimalist perspective, the empty set is not a vector space since there is no zero vector. However the set $V = \{0\}$ with the obvious rules of addition and scalar multiplication is a vector space.

Let $M_{m,n}(F)$ be the set of $m \times n$ matrices, with entries in $F$. It is straightforward to check that with the standard rules for addition and scalar multiplication, $M_{m,n}(F)$ is a vector space over $F$. Flattening out a matrix in an obvious way, this vector space is really the same as the vector space $F^{mn}$. The polynomials $P_d(F)$ of degree at most $d$ form a vector space, with the usual rules for addition and scalar multiplication.

For a highly non-trivial example of a vector space, let $\mathbb{R}^{[0,1]}$ be the set of all functions from the interval $[0,1]$ to the field $\mathbb{R}$. Given

$$f: [0, 1] \longrightarrow \mathbb{R} \quad \text{and} \quad g: [0, 1] \longrightarrow \mathbb{R},$$

and $\lambda \in \mathbb{R}$ define

$$f + g: [0, 1] \longrightarrow \mathbb{R} \quad \text{and} \quad \lambda f: [0, 1] \longrightarrow \mathbb{R}$$

by the rules $(f + g)(x) = f(x) + g(x)$ and $(\lambda f)(x) = \lambda f(x)$. It is not hard to see that $\mathbb{R}^{[0,1]}$ is a vector space. In fact we get a similar example if we look at the set $F^X$ of all functions from the set $X$ into a field $F$. This set is naturally a vector space, with pointwise addition and scalar multiplication.

We are going to redefine what it means to be a subspace:

**Definition 7.2.** Let $V$ be a vector space over a field $F$. A subset $W$ is called a **subspace** if $W$, together with the rules for addition and scalar multiplication inherited from $V$, becomes a vector space.

The only subtlety with this definition is the meaning of inherited. By assumption $V$ comes with a rule for addition and scalar multiplication. These are two functions

$$+ = a: V \times V \longrightarrow V \quad \text{and} \quad \cdot = s: F \times V \longrightarrow V.$$  

Inherited means simply that we restrict these functions to $W$,

$$b: W \times W \longrightarrow V \quad \text{and} \quad t: F \times W \longrightarrow V.$$ 

The problem is that the target is not right. We should land up in $W$ but instead we arrive in $V$. However we will land up in $W$ provided that $W$ is closed under addition and scalar multiplication.

In fact the two definition of subspace are easily seen to be compatible:

**Lemma 7.3.** Let $V$ be a vector space. A subset $W$ is a subspace if and only if
(1) $0 \in W$ (or even just $W$ is non-empty).
(2) $W$ is closed under addition.
(3) $W$ is closed under scalar multiplication.

**Proof.** Suppose that $W$ is a subspace. Then $W$ contains a vector $z$, which acts as the zero vector. Consider the equality

$$z + z = z.$$ 

It holds as $z$ is the zero vector in $W$. But we could read this equation as taking place in $V$. Subtracting $z$ from both sides, we get $z = 0$. So the zero vector in $W$ is in fact the zero vector in $V$. Hence (1). (2) and (3) are implicit in the meaning of inherited, as explained above.

Now suppose that $W$ satisfies (1)-(3). (2) and (3) say that addition and scalar multiplication in $V$ induce well-defined operations of addition and scalar multiplication in $W$. If $w \in W$ then $0 = 0 \cdot w \in W$ as $W$ is closed under scalar multiplication. In particular $W$ contains a zero vector. Given $w \in W$, $-1 \cdot w \in W$ as $W$ is closed under scalar multiplication. But it is easy to see that $-w = -1 \cdot w$ is the additive inverse of $w$. The rest of the axioms follow automatically, since they hold in $V$ and the rest of the axioms don’t involve existential quantifiers (a fancy way of saying they don’t say “there exists . . .”) only universal quantifiers (“for all . . .”). □

As before, $\{0\}$ are $V$ are two trivial examples of subspaces. We have already seen that the span of any finite set and the solutions of homogeneous equations always give subspaces of $F^d$. The set of upper triangular matrices in $M_{n,n}(F)$ is a vector subspace. Indeed, 0 is an upper triangular matrix, the sum of two upper triangular matrices is upper triangular and a scalar multiple of upper triangular is upper triangular. Suppose that $m$ and $n > 1$. Then the set of matrices of rank at most one is not a subspace. Indeed, if $m = n = 2$ then consider

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. $$

Both matrices on the LHS have rank one but their sum, the matrix on the RHS has rank two. Thus the set of matrices of rank at most one is not closed under addition. Similar examples pertain for any $m$ and $n > 1$. Note however that the zero matrix has rank zero and a scalar multiple of a rank one matrix is rank one.

$$P_0(F) \subset P_1(F) \subset P_2(F) \subset \cdots \subset P(F),$$

where $P(F)$ denotes the vector space of all polynomials with coefficients in $F$. Actually there is nothing to check, this follows from the definitions (in other words to add two polynomials of degree at most
three, it does not matter if we consider them as polynomials of degree at most five, the rules for addition won’t change).

However the set of polynomials of degree \( d > 0 \) is not a vector subspace of the polynomials of degree at most \( d \). For example

\[
x + (1 - x) = 1.
\]

Both polynomials on the LHS have degree one, but their sum on the RHS has degree zero. Even quicker the zero polynomial does not have degree \( d \). Similarly monic polynomials of degree \( d \) do not form a subspace.

\[
x + (1 + x) = 2x + 1.
\]

Both polynomials on the LHS are monic, but the polynomial on the right is not. Also note that

\[
2(x) = 2x.
\]

Again even quicker, zero is not a monic polynomial.

The subset of continuous functions

\[
C[0, 1] \subset \mathbb{R}^{[0, 1]},
\]

is a vector subspace. The the zero function is continuous, the sum of two continuous functions is continuous and a scalar multiple of a continuous function is continuous.