Practice Midterm Solutions

1. Suppose that $f$ is continuous and $2\pi$-periodic. Is it true that

$$\lim_{N \to \infty} \int_{0}^{2\pi} |S_N f(x) - f(x)| \, dx = 0?$$

Explain your answer.

Yes. We know from a theorem in the course that $\|S_N f - f\| \to 0$, and so

$$\int_{0}^{2\pi} |S_N f - f|^2 \to 0.$$

Now by Cauchy-Schwarz

$$\int_{0}^{2\pi} |S_N f(x) - f(x)| \, dx = \int_{0}^{2\pi} |S_N f(x) - f(x)| \cdot 1 \, dx \leq \left( \int_{0}^{2\pi} |S_N f(x) - f(x)|^2 \, dx \right)^{1/2} \cdot (2\pi)^{1/2} \to 0.$$

2. Let $g_N$ be the $2\pi$-periodic function defined on $[-\pi, \pi]$ by setting $g_N(x) = \pi N$ if $|x| \leq 1/N$ and $g_N(x) = 1/N$ if $1/N \leq |x| \leq \pi$. Suppose that $f$ is a $C^0$ and $2\pi$-periodic function. Prove from first principles that $\lim_{N \to \infty} f \ast g_N(0) = f(0)$.

“Prove from first principles” means that you cannot cite the good kernel theorem, but you can imitate the proof of the good kernel theorem.

By the definition of a convolution,

$$f \ast g_N(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) g_N(-y) \, dy = \frac{1}{2\pi} \int_{-1/N}^{1/N} \pi N f(y) dy + \frac{1}{2\pi} \int_{1/N \leq |y| \leq \pi} (1/N) f(y) dy.$$

We will show that the first integral tends to $f(0)$ and that the second integral tends to 0.

Since $f$ is $C^0$, for any $\epsilon > 0$, there is an $N_\epsilon$ so that for all $N \geq N_\epsilon$, for all $|y| \leq 1/N$, $|f(0) - f(y)| < \epsilon$. We also note that $\frac{1}{2\pi} \int_{-1/N}^{1/N} \pi N = 1$. So if $N > N_\epsilon$, we see that

$$\left| \frac{1}{2\pi} \int_{-1/N}^{1/N} \pi N f(y) dy - f(0) \right| \leq \frac{1}{2\pi} \int_{-1/N}^{1/N} |\pi N (f(y) - f(0))| \, dy \leq \frac{1}{2\pi} \int_{-1/N}^{1/N} \pi N \epsilon \, dy = \epsilon.$$
Since $\epsilon > 0$ was arbitrary, this shows that $\frac{1}{2\pi} \int_{-\frac{1}{N}}^{\frac{1}{N}} \pi N f(y)dy \to f(0)$.

On the other hand,

$$\left| \frac{1}{2\pi} \int_{1/N \leq |y| \leq \pi} (1/N)f(y)dy \right| \leq \max |f| \cdot (1/N) \to 0.$$

3. This problem is about the partial differential equation $\partial_t u(x,t) = \partial_x^3 u(x,t)$ (where $x \in \mathbb{R}$).

a.) Consider the initial value problem $\partial_t u(x,t) = \partial_x^3 u(x,t)$, with initial data $u(x,0) = f(x)$, where $f$ is a Schwartz function. Give a formula for the solution $u(x,t)$ in terms of $\hat{f}$. You don’t need to give a full proof that the formula holds.

Solution to a.)

First we rewrite the equation in terms of $\hat{u}$:

$$\partial_t \hat{u}(\omega, t) = (2\pi i \omega)^3 \hat{u}(\omega, t).$$

We have the initial values $\hat{u}(\omega,0) = \hat{f}(\omega)$. Solving the resulting ODE gives

$$\hat{u}(\omega, t) = e^{(2\pi i \omega)^3 t} \hat{f}(\omega) = e^{-8\pi^3 i \omega^3 t} \hat{f}(\omega).$$

Now Fourier inversion gives

$$u(x,t) = \int_{\mathbb{R}} e^{2\pi i \omega x} e^{-8\pi^3 i \omega^3 t} \hat{f}(\omega) d\omega. \quad (*)$$

b.) Using the formula from part a.), prove the following estimate. Suppose that

- $\int_{\mathbb{R}} |f(x)|dx \leq 1$,
- $\int_{\mathbb{R}} |\partial_x^2 f(x)|dx \leq 1$.

Prove that $|u(x,t)| \leq 100$ for all $x \in \mathbb{R}$ and all $t \geq 0$.

Solution to b.): First we use our two inequalities about $f$ to estimate $\hat{f}(\omega)$. For any function $g$, the triangle inequality gives

$$|\hat{g}(\omega)| = \left| \int_{\mathbb{R}} e^{-2\pi i \omega x} g(x)dx \right| \leq \int_{\mathbb{R}} |g(x)|dx.$$

In particular, the first inequality gives immediately $|\hat{f}(\omega)| \leq 1$ for all $\omega \in \mathbb{R}$.

Let $f_2 := \partial_x^2 f$. As on the formula sheet, we know that $\hat{f_2}(\omega) = (2\pi i \omega)^2 \hat{f}(\omega)$. Since $\int_{\mathbb{R}} |f_2(x)|dx \leq 1$, we see that

$$\left| (2\pi i \omega)^2 \hat{f}(\omega) \right| \leq 1,$$
and so

\[ |\hat{f}(\omega)| \leq \frac{1}{4\pi^2\omega^2}. \]

Combining our two bounds on \( \hat{f}(\omega) \), we see that

\[ |\hat{f}(\omega)| \leq \min\left(1, \frac{1}{4\pi^2\omega^2}\right) \leq \frac{10}{1 + |\omega|^2}. \]

Finally, we insert this information into the formula (\( * \)) for the solution \( u \):

\[ |u(x, t)| = \left| \int_{\mathbb{R}} e^{2\pi i\omega x} e^{-8\pi^3 i\omega t} \hat{f}(\omega) d\omega \right| \leq \int_{\mathbb{R}} |\hat{f}(\omega)| d\omega \leq \int_{\mathbb{R}} \frac{10}{1 + |\omega|^2} d\omega \leq 100. \]

4. Suppose that \( g(x) = 1 + \cos x \). In this problem, we consider what happens when we convolve \( g \) with itself many times. Here we use convolution in the setting of periodic functions, so

\[ f_1 * f_2(x) := \frac{1}{2\pi} \int_0^{2\pi} f_1(y) f_2(x - y) dy. \]

Let \( g_2 := g * g \). Let \( g_{k+1} := g * g_k \). Find \( \lim_{k \to \infty} g_k(1) \).

Hint: Recall that if \( f_3 = f_1 * f_2 \), then \( \hat{f}_3(n) = \hat{f}_1(n) \hat{f}_2(n) \).

We know that \( \hat{g}_k(n) = (\hat{g}(n))^k \). Now \( g(x) = 1 + \cos x = 1 + (1/2)e^{ix} + (1/2)e^{-ix} \).

Therefore, \( \hat{g}(0) = 1, \hat{g}(\pm 1) = (1/2) \) and \( \hat{g}(n) = 0 \) for other \( n \). Therefore, \( \hat{g}_k(0) = 1, \hat{g}_k(\pm 1) = 2^{-k} \), and \( \hat{g}(n) = 0 \) for other \( n \). Since \( g_k \) is clearly smooth, it is equal to the sum of its Fourier series, and we get the simple formula

\[ g_k = 1 + 2^{-k}e^{ix} + 2^{-k}e^{-ix} = 1 + 2^{-k} \cos x. \]

Therefore, \( \lim_{k \to \infty} g_k(1) = 1. \)

5. Suppose that \( f \) is \( C^2 \) and \( 2\pi \)-periodic. Suppose that \( \int_0^{2\pi} |f'(x)|^2 = 1 \). Prove that \( |S_N f(x) - f(x)| \leq 10N^{-1/2} \).

Since \( f \) is \( C^2 \) periodic, we know that \( S_N f(x) \to f(x) \). Therefore,

\[ |S_N f(x) - f(x)| \leq \sum_{|n| > N} |\hat{f}(n)|. \]

We let \( g := f' \). On the other hand, by Parseval’s theorem, we know that
1 \geq \|g\|^2 = \sum_n |\hat{g}(n)|^2 = \sum_{n=-\infty}^{\infty} |n|^2 |\hat{f}(n)|^2.

We will use this inequality and Cauchy-Schwarz to bound \(\sum_{|n| > N} |\hat{f}(n)|\):

\[\sum_{|n| > N} |\hat{f}(n)| = \sum_{|n| > N} |\hat{f}(n)||n||n|^{-1} \leq \left( \sum_n |n|^2 |\hat{f}(n)|^2 \right)^{1/2} \left( \sum_{|n| > N} |n|^{-2} \right)^{1/2} \leq 1 \cdot (10N^{-1})^{1/2} \leq 10N^{-1/2}.\]